Noncommutative Geometric Invariants of Fomin-Kirillov Algebras and their Generalizations

Aditya Saini

Advisor: Dr. Be'eri Greenfeld
Faculty Advisor: Dr. Danna Zhang

Department of Mathematics

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Contents

1 Introduction

We study an important class of noncommutative algebras namely Fomin-Kirillov algebra and their subalgebras, through a geometric prism. An important concept in noncommutative projective algebraic geometry is of *point modules*, which substitute points in classical algebraic geometry. A key fact is that one can encode module actions into points in certain projective varieties, where the relations of the algebra translate to (multilinearized) homogeneous polynomial constraints. In this thesis, we study the spaces of truncated point modules over these algebras.

Our focus is on the truncated point modules over subalgebras of Fomin-Kirillov algebras associated with certain graph structures. We establish that Fomin-Kirillov algebras do not admit any non-trivial truncated point modules. Furthermore, for generalized Fomin-Kirillov algebras associated with trees, we show that the highest degree of a truncated point module is at most the number of edges of that tree, and this bound is sharp, as we demonstrate through concrete example. We start by studying two important classes – path graphs and star graphs – and then use the structure of line graphs to reduce the general case to these classes, proving our main result.

We also explore several other examples and compute their associated spaces of point modules. We utilize concepts from noncommutative algebra, noncommutative projective geometry and graph theory.

Our findings build a bridge between combinatorial graph theory and the algebraic and geometric properties of (generalized) Fomin-Kirillov algebras. These results provide insights into the algebraic and geometric properties of these algebras, with implications for broader applications in noncommutative projective geometry. The study mainly focuses on truncated point modules of generalized Fomin-Kirillov algebras being as restricted as possible.

We start by defining key terms from noncommutative algebra and then review the correspondence between truncated point modules and points on projective varieties. We then compute truncated point modules over Fomin-Kirillov algebra. After that, we turn to study truncated point modules over generalized Fomin-Kirillov algebras and prove the main results.

This thesis is partially based on the author's paper [1], the thesis provides more results and examples and a wider background.

2 Preliminaries and Definitions

2.1 Graded algebras

Let k be a field. An **algebra over** k, or a k-algebra, is a ring with a vector space structure compatible with the ring structure. There is also a well defined scalar multiplication. Namely, $(A, +, \cdot)$ is a ring and, at the same time, $(A, +)$ is a vector space over k.

A **graded ring** is an algebraic structure that can be decomposed into a direct sum of abelian groups or modules, indexed by a set of integers. More formally, a graded ring A is a ring that can be expressed as:

$$A = \bigoplus_{i=0}^{\infty} A_i$$

where each A_i is an abelian group, and the multiplication in the ring satisfies the property:

$$A_i \cdot A_j \subseteq A_{i+j}$$

This means that when you multiply elements from the graded components A_i and A_j, the result lies in the component A_{i+j}. Each A_i is referred to as the degree i component of the ring.

A graded ring is said to be **connected** if the component A_0 is equal to the base field k. That is, the lowest-degree component is the field k itself:

$$A_0 = k$$

In this context, we assume that the algebra contains a base field k. The connected graded property ensures that the graded ring has a natural starting point at degree 0, which simplifies the algebraic structure.

We say that a graded ring A as being **generated in degree 1** if the entire ring A can be generated by the elements in the degree 1 component, denoted by A_1. In other words, every element in the graded ring can be expressed as a combination of products of elements from A_1 and the base field k: A is generated as an algebra by A_1. This implies that all higher-degree components A_i for $i > 1$ can be written in terms of products of elements in A_1.

Example 1. Polynomial rings.

One common example of a graded ring is the (commutative) polynomial ring $k[x_1, \ldots, x_d]$, where the grading is given by the total degree of each monomial and the degree of each variable $x_1, \ldots, x_d$ is 1. (We can even set the degrees of $x_1, \ldots, x_d$ to be any natural number but then the algebra will not be generated in degree 1 if not all are set to have degree 1.) In this case, the algebra can be expressed as:

$$A = \bigoplus_{i=0}^{\infty} A_i$$

where A_i is the span of all monomials of degree i. For instance, if $A = k[x_1, x_2]$, the degree 0 component A_0 is k, and the degree 1 component A_1 is the span of x_1 and x_2. The degree 2 component A_2 would then be the span of x_1^2, x_1x_2, x_2^2, and so on.

Thus, for a graded ring generated by $x_1, x_2, \ldots, x_d$, each A_i consists of all monomials of total degree i, and the dimension of each component A_i over k can be determined by counting the number of distinct monomials of degree i. The number of independent monomials of degree i in the polynomial ring in d variables gives:

$$\dim_k A_i = \binom{d+i-1}{i-1}$$

We could relate this with distributing i balls (number of variables in a monomial of degree i) to d bins (variables).

Example 2. Quantum polynomial rings.

A more general setting involves **quantum polynomial rings**, where the relations between the variables are twisted by scalar factors. For example, consider a quantum polynomial ring where the variables x_1, x_2 satisfy the relation:

$$x_1x_2 = qx_2x_1$$

for some nonzero scalar $q \in k^\times$. This defines a noncommutative graded ring, with the multiplication in the ring being governed by these quantum relations. The grading still satisfies the condition that multiplying elements of degree i and degree j yields an element of degree $i+j$, but the algebra is no longer commutative. There are also versions of quantum polynomial rings with more variables and with more quantum parameters.

2.2 Graded Modules

A **graded module** is a special kind of module that has grading comatible with the grading of a ring. Let A be a graded ring (respectively, algebra) and M a left A-module. We say that M is **graded** if it can be decomposed as a direct sum of abelian groups (resp. vector subspaces), indexed by non-negative integers:

$$M = \bigoplus_{i=0}^{\infty} M_i$$

where each M_i is called the **degree i component** of M. The action of the ring A on the module M respects the grading. This means that if $a \in A_i$ and $m \in M_j$, then the action of a on m must lie in the $(i+j)$-th degree component of M:

$$A_i \cdot M_j \subseteq M_{i+j}$$

Thus, the grading of the module is compatible with the grading of the ring.

Example 3. Every graded ring (or algebra) A is a graded module over itself.

Example 4. Let A be a connected graded k algebra, let M be a graded module given by $M_0 = k$ and $M_i = 0$ for all positive i. We call this the trivial module. The module action is given as follows: if $a \in A_i$, $m \in M_j$ then $a \cdot m = 0$ if $i + j > 0$, and if $a \in A_0 = k$ and $m \in M_0 = k$, then $a \cdot m$ is given by the standard multiplication in the base field.

2.2.1 Module homomorphisms and cyclic modules

Suppose that A is a k-algebra. A **module homomorphism** $\varphi\colon N \to M$ is a linear map that preserves the A-module structure, namely, $\varphi(a \cdot n) = a \cdot \varphi(n)$ for all $a \in A$ and $n \in N$. This homomorphism is surjective if the image of φ is all of M, and injective if the kernel of φ is trivial, i.e., the set of all elements $n \in N$ such that $\varphi(n) = 0$ consists only of zero.

A module M is called **cyclic** if it can be generated by a single element $m \in M$. Consider the A-module homomorphism $\varphi\colon A \to M$ given by:

$$\varphi(a) = a \cdot m$$

where $a \in A$ and $m \in M$. In this case, the module M is isomorphic to the quotient of the graded ring A by a left ideal L:

$$M \cong A/L$$

where L is a left ideal of A that consists of all elements $a \in A$ such that $a \cdot m = 0$. In other words:

$$L = \{a \in A \mid a \cdot m = 0\}.$$

This gives a construction where the cyclic module is determined by the graded ring and the left ideal that annihilates the generator m.

In the next section we will point modules that are an important case of cyclic modules.

2.2.2 Point modules and their truncations

A **point module** is a special kind of graded module that plays an important role in noncommutative algebra and algebraic geometry. Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a graded algebra. A graded A-module M is called a **point module** if:

- M is cyclic, i.e., M is generated as a module by a single element from M_0.
- The dimension of each graded component M_i over k is 1:

$$\dim_k M_i = 1 \text{ for all } i \geq 0.$$

Thus, the module M has the form:

$$M = \bigoplus_{i=0}^{\infty} M_i$$

with each M_i being a 1-dimensional vector space over k.

Example 5. Polynomial ring as a point module.

Let $A = k[x]$, the polynomial ring in one variable. The point module M is given by:

$$M = A = k \oplus kx \oplus kx^2 \oplus \dots$$

Here, $M_0 = k$, $M_1 = kx$, $M_2 = kx^2$, and so on. Each component M_i is spanned by the unique monomial of degree i.

Example 6. Bivariate polynomial rings and their point modules.

Let $A = k[x, y]$ and consider the point module:

$$M = \frac{k[x, y]}{\langle x - \lambda y \rangle} = k \oplus ky \oplus ky^2 \oplus \dots,$$

where $\lambda \in k$ and the relation $x = \lambda y$ is imposed. In this case, the components $M_0, M_1, M_2, \dots$ are spanned by powers of y.

Another example of a point module over A is $M = k[x, y]/\langle y \rangle = k \oplus kx \oplus kx^2 \oplus \dots$ (and these are all of the point modules over A).

2.2.3 Truncated point modules

A truncated point module of degree d is a finite-dimensional version of a point module:

$$M = \bigoplus_{i=0}^{d} M_i$$

with:

- $M_0 = k$
- $\dim_k M_i = 1$ for $0 \leq i \leq d$.
- $M_i = 0$ for all $i > d$.

So it is similar to a point module, but 'stops' after degree d.

Example 7. Truncated point modules of degrees $0, 1$.

A truncated point module of degree 0 is trivial shown in Example 4. A truncated point module of degree 1 looks like $M = ke_0 \oplus ke_1$. For every $a \in A_1$ we have $a \cdot e_0 = \lambda_a e_1$ for some scalar $\lambda_a \in k$, and at least one of the λ_a's is non-zero. Otherwise, the module is not going to be generated by its degree 0 component, and will not be cyclic.

2.3 Spaces of point modules

2.3.1 Projective spaces

Let k be a field, the **projective space** $\mathbb{P}^d$ **over** k is defined as the set of equivalence classes of $(d+1)$-tuples $(x_0, \dots, x_d) \in k^{d+1}$ $\{0\}$ under the equivalence relation: $(x_0, \dots, x_d) \sim (\lambda x_0, \cdots, \lambda x_d)$ for all $\lambda \in k^\times$. We denote the equivalence class of $(x_0, \dots, x_d)$ by $[x_0 : \cdots : x_d]$.

Example 8. The projective line $\mathbb{P}^1$.

We have the following representatives:

1. $[1:\alpha] \sim [1:\beta]$ if and only if $\alpha = \beta$. We think of $[1\colon \alpha]$ as $\frac{\alpha}{1} = \alpha \in k$.
2. $[1:\alpha] \not\sim [0:1]$ for any α. We think of $[0\colon 1]$ as $\frac{1}{0} = \infty$.

To check the non-equivalence of $[1:\alpha]$ and $[1:\beta]$, we see that is $(1,\alpha) \sim (1,\beta)$ then there must exist some scalar $c \in k^\times$ such that $1 = c \cdot 1$ and $\alpha = c \cdot \beta$ we conclude that $c = 1$ and get $\alpha = \beta$. Next, no point of the form $[1:\alpha]$ is equivalent to $[0:1]$. For $[1:\alpha]$ and $[0:1]$ to be equivalent, there must exist some scalar $c \in k^\times$ such that $(1,\alpha) \sim (0,1)$ which gives $1 = c \cdot 0$, a contradiction.

Therefore we can think of $\mathbb{P}^1$ as $k \cup \{\infty\}$.

2.3.2 Parametrizing (truncated) point modules

The Action of the Generators

Let A be a connected graded algebra generated in degree 1 and suppose $A \cong k\langle x_1, \dots, x_d\rangle / \langle f_1, \dots\rangle$, a free algebra modulo homogeneous relations. A point module can be written as:

$$M = ke_0 \oplus ke_1 \oplus ke_2 \oplus \dots$$

where each e_i spans the i-th graded component M_i. The action of the algebra's generators $x_j,\ 1 \le j \le d$ on the module is given by:

$$\underbrace{x_j}_{\text{degree } 1} \cdot \underbrace{e_i}_{\text{degree } i} = \underbrace{\lambda^i_j e_{i+1}}_{\text{degree } i+1} .$$

Remark: For each i, at least one of the $\lambda^i_1, \dots, \lambda^i_d$ is non-zero, otherwise we will not be able to get e_{i+1} from e_0, and therefore M will not be generated by e_0. Therefore, we may regard $[\lambda^i_1\colon \cdots : \lambda^i_d]$ as a point in $\mathbb{P}^{d-1}$.

When are two point modules isomorphic?

Suppose we have another A-point module

$$M' = ke'_0 \oplus ke'_1 \oplus \cdots$$

where the action of the algebra's generators is given by

$$x_j \cdot e'_i = \mu^i_j e'_{i+1}.$$

for some $\mu^i_j \in k$.

Then $M \cong M'$ if and only if for every $i \ge 0$, we have $[\lambda^i_1 : \cdots : \lambda^i_d] = [\mu^i_1 : \cdots : \mu^i_d]$ as points in $\mathbb{P}^{d-1}$.

Point Modules and Subsets of $\mathbb{P}^{d-1} \times \mathbb{P}^{d-1} \times \cdots$.

We begin by considering point modules over the free algebra $k\langle x_1, \dots, x_d\rangle$. A point module over the free algebra $k\langle x_1, \dots, x_d\rangle$ can be described using the sequences of scalars λ^j_i that satisfy:

$$x_j \cdot e_i = \lambda^i_j e_{i+1}.$$

and by the above discussion, there is a one-to-one correspondence between point modules and sequences in the product of projective spaces:

$$\mathbb{P}^{d-1} \times \mathbb{P}^{d-1} \times \cdots$$

assigning the above module to the sequence

$$([\lambda_1^0 : \cdots : \lambda_d^0], [\lambda_1^1 : \cdots : \lambda_d^1], [\lambda_1^2 : \cdots : \lambda_d^2], \dots).$$

Similarly, truncated point modules of degree n correspond to sequences in $\mathbb{P}^{d-1} \times \underbrace{\cdots}_{m \text{ times}} \times \mathbb{P}^{d-1}$.

Now, let us introduce some non-trivial relations into the algebra. Consider an algebra A obtained by quotienting the free algebra by a set of relations:

$$A = k\langle x_1, x_2, \dots, x_d \rangle / \langle f_1, f_2, \dots \rangle,$$

where the f_i's represent the relations.

For an n-truncated point module M over A we can fix a homogeneous basis $M = ke_0 \oplus \cdots \oplus ke_n$ and encode the action of A on M by constants $\lambda_i^j \in k,\ 1 \leq j \leq d,\ 0 \leq i \leq n-1$ such that:

$$x_j \cdot e_i = \lambda_j^i e_{i+1}.$$

Thus $([\lambda_1^0 \colon \cdots : \lambda_d^0], \dots, [\lambda_1^{n-1} \colon \cdots : \lambda_d^{n-1}]) \in \mathbb{P}^{d-1} \times \cdots \times \mathbb{P}^{d-1}$. Those points in the product of projective spaces corresponding to truncated point modules can be found as the solutions of a concrete system of homogeneous equations derived from the presentation $k\left\langle x_1, \dots, x_d \right\rangle / \left\langle f_1, f_2, \dots \right\rangle$ by multilinearization, namely, each relation $f = \sum_I c_I x_{i_1} \cdots x_{i_t}$ induces the polynomial equations $\sum_I c_I \lambda_{i_1}^{s+t-1} \cdots \lambda_{i_t}^{s} = 0$ for each $0 \leq s \leq n-t$. The set of points in $\left(\mathbb{P}^{d-1}\right)^{\times n}$ solving these multilinearized polynomial equations is in bijection with $\mathcal{P}_n(A)$, the set of n-truncated point modules over A; see [3, Proposition 3.9] and [12].

Under these relations, the sequences of λ_i^j's are restricted. For example, if $x_1^2 = 0$, then:

$$x_1^2 \cdot e_i = 0 \cdot e_i = 0,$$

which imposes the condition:

$$\lambda_1^{i+1} \lambda_1^i = 0.$$

This means that for each i, at least one of λ_1^i or λ_1^{i+1} must be zero.

Example 9. Polynomial rings

Recall $k[x_1, \dots, x_d] \cong k\langle x_1, \dots, x_d \rangle / \langle x_i x_j - x_j x_i \text{ for all } 1 \leq i < j \leq d \rangle$. Let us focus on the case $d = 2$ first. Consider the following equalities:

$$x_1 x_2 \cdot e_i = x_2 x_1 \cdot e_i$$

$$x_1 \cdot \lambda_2^i \cdot e_{i+1} = x_2 \cdot \lambda_1^i \cdot e_{i+1}$$

$$\lambda_2^i x_1 \cdot e_{i+1} = \lambda_1^i x_2 \cdot e_{i+1}$$

$$\lambda_2^i \lambda_1^{i+1} \cdot e_{i+2} = \lambda_1^i \lambda_2^{i+1} \cdot e_{i+2}$$

which implies that

$$\lambda_2^i \lambda_1^{i+1} = \lambda_1^i \lambda_2^{i+1}.$$

From this, we obtain the ratio

$$\frac{\lambda_1^{i+1}}{\lambda_2^{i+1}} = \frac{\lambda_1^i}{\lambda_2^i} = \alpha, \text{ for all } i$$

Thus, $[\lambda_1^i \colon \lambda_2^i] = [\lambda_1^{i+1} \colon \lambda_2^{i+1}]$ for all i. Conclusion: The subset of $\mathbb{P}^1 \times \mathbb{P}^1 \times \cdots$ corresponding to point modules of $k[x_1, x_2] \cong k\langle x_1, x_2\rangle / \langle x_1x_2 - x_2x_1\rangle$ is $\{([\lambda_1^0 : \lambda_2^0], [\lambda_1^0 : \lambda_2^0], [\lambda_1^0 : \lambda_2^0], \dots) | [\lambda_1^0 : \lambda_2^0] \in \mathbb{P}^1\}$, which is in bijection with $\mathbb{P}^1$. This extends to multivariate polynomial rings, replacing $\mathbb{P}^1$ by $\mathbb{P}^{d-1}$. Likewise, $\mathcal{P}_n(k[x_1, \dots, x_d]) \cong \mathbb{P}^{d-1}$ for all n.

Example 10. The algebra $k\langle x_1, x_2\rangle / \langle x_1x_2\rangle$.

The point modules over A correspond to a subset of the product of projective spaces:

$$\left\{ \left(\underbrace{[\lambda_1^0 : \lambda_2^0]}_{P_0}, \underbrace{[\lambda_1^1 : \lambda_2^1]}_{P_1}, \underbrace{[\lambda_1^2 : \lambda_2^2]}_{P_2}, \dots \right) \mid \lambda_1^{i+1}\lambda_1^i = 0 \right\} \subseteq \mathbb{P}^{d-1} \times \mathbb{P}^{d-1} \times \cdots$$

with constrains given by multilinearizing the relations of A. The relation $x_1x_2 \cdot e_i = 0$ gives $x_1 \cdot \lambda_2^i e_{i+1} = 0$, so $\lambda_2^i x_1 \cdot e_{i+1} = 0$, so $\lambda_2^i \lambda_1^{i+1} \cdot e_{i+2} = 0$. For all i:

$$\lambda_1^{i+1} \lambda_2^i = 0$$

From these conditions, we obtain:

$$\lambda_1^{i+1}\lambda_2^i = 0 \implies \text{Either } \lambda_2^i = 0 \text{ or } \lambda_1^{i+1} = 0.$$

We consider two cases:

- If $\lambda_2^i = 0$, then $P_i = [1 : 0] = \frac{0}{1} = 0$.
- If $\lambda_1^{i+1} = 0$, then $P_{i+1} = [0 : 1] = \frac{1}{0} = \infty$.

Thus, for every i, either $P_i = \infty$ or $P_{i+1} = 0$. Conclusion:

The subset of $\mathbb{P}^1 \times \mathbb{P}^1 \times \cdots$ corresponding to point modules of $k\langle x_1, x_2\rangle / \langle x_1, x_2\rangle$ is the union $L_0 \cup L_1 \cup \cdots$ where:

$$\begin{array}{lcll} L_0 & : & \mathbb{P}^1 \times \{0\} \times \{0\} \times \cdots & (\text{e.g.: } (\alpha, 0, 0, \dots)) \\ L_1 & : & \{0\} \times \mathbb{P}^1 \times \{0\} \times \cdots & (\text{e.g.: } (\infty, \alpha, 0, \dots)) \\ L_2 & : & \{0\} \times \{0\} \times \mathbb{P}^1 \times \cdots & (\text{e.g.: } (\infty, \infty, \alpha, \dots)) \\ \vdots & & & \end{array}$$

This is a countably infinite union of projective lines.

3 Fomin-Kirillov algebras and their modules

3.1 Definition and Literature

Fomin-Kirillov algebras

Fomin-Kirillov algebras $\mathcal{FK}_n,\ n = 3, 4, 5, \ldots$, introduced by Fomin and Kirillov in 1999 in their study of the cohomology of flag manifolds [2], are noncommutative quadratic algebras which are intimately related to cohomology theory and homological algebra, algebraic combinatorics, Schubert calculus, Nichols algebras, Hopf algebras, and more [4, 5, 7, 8, 9, 10, 11, 13, 15].

- In [4], Jonah Blasaik, Ricky Ini Lui and Karola Meszaros explored the structural parallels of $\mathcal{FK}_n$ with Coxeter groups and nil-Coxeter algebras by studying the subalgebras $\mathcal{FK}_G$ associated with graphs G. The study involves determining explicit bases, Hilbert series, and conjectures, particularly for finite-dimensional cases related to Dynkin diagrams and cycles.

- In [5], Fomin and Procesi introduce Hopf-algebraic tools for studying the quadratic associative algebras $\mathcal{FK}_n$. By defining a Hopf algebra structure, it explores tensor product decompositions and the implications for the Hilbert series factorization conjectured by Kirillov.

- In [7], Heckenberger and L. Vendramin use $\mathcal{FK}_3$ to focus on PBW deformations of graded algebras. They studied plynomial identities of $\mathcal{FK}_3$ using PBW deformation.

- In [8], Anatol N. Kirillov and Toshiaki Maeno explore the application of $\mathcal{FK}_n$ to root systems of type A, generalizes equivariant Pieri rules for Schubert polynomials, and establishes connections with Nichols-Woronowicz algebras for finite Coxeter systems, thereby expanding the theoretical framework of algebraic and geometric structures.

- In [9], Cristian Lenart extends the study of Fomin-Kirillov algebras by developing a new approach to the multiplication of Schubert classes in the K-theory of flag varieties. By defining K-theoretic versions of the Dunkl elements introduced by Fomin and Kirillov, it demonstrates their commutativity and uses these elements to describe the structure constants of K -theory in terms of Schubert classes.

- In [10], this study by Karola Meszaros, Greta Panova and Alexander Postnikov addresses the multiplication of Schubert polynomials with Schur polynomials and establishes explicit combinatorial rules for expansion coefficients in specific cases. By leveraging the Dunkl elements of the Fomin-Kirillov algebra, it proves several special cases of their nonnegativity conjecture. This approach not only confirms insights from $\mathcal{FK}_n$ but also extends to the quantum cohomology ring of the flag manifold and Gromov-Witten invariants, offering significant progress in developing combinatorial rules for these coefficients in enumerative algebraic geometry.

- In [11], Alexander Milinski and Hans-Jurgen Schneider study link-indecomposable Hopf algebras associated with Coxeter groups and their relations to Nichols algebras, focusing on finite-dimensional cases. It highlights the Fomin-Kirillov algebras $\mathcal{FK}_n$ as braided Hopf algebras over symmetric groups, examining their Hilbert series divisibility properties and freeness relations. These insights connect the quadratic algebras $\mathcal{FK}_n$ to broader Hopf algebra theory, advancing understanding of their role in cohomology and symmetric structures.

- In [13], Dragos Stefan and Cristian Vay examine the 12-dimensional Fomin-Kirillov algebra $\mathcal{FK}_3$ identified as isomorphic to the Nichols algebra over the symmetric group $\S_3$

- In [15], Chelsea Walton and James J. Zhang study ring-theortic and homological properties of the quadratic dual of the Fomin Kirillov algebras $\mathcal{FK}_n$

Fomin Kirillov algebras $\mathcal{FK}_n$ could be defined as the algebra with generators $x_{(ij)}$, where $i, j \in \{1, \ldots, n\}$, and relations:

$$x_{(ij)} + x_{(ji)} = 0,$$

$$x_{(ij)}^2 = 0,$$

$$x_{(ij)}x_{(jk)}x_{(ki)} + x_{(jk)}x_{(ki)}x_{(ij)} + x_{(ki)}x_{(ij)}x_{(jk)} = 0,$$

$$x_{(ij)}x_{(kl)} = x_{(kl)}x_{(ij)},$$

for any distinct i, j, k, l [14]. Because of the first relation we can write that $x_{(ij)}$ = -$x_{(ji)}$, so therefore we define the relations with $i < j$ and in the following way: Let us focus on the Fomin-Kirillov algebra $\mathcal{FK}_n$ for some n. It has the following presentation: $\mathcal{FK}_n$ is generated by $\{x_{ij} \mid 1 \leq i < j \leq n\}$ (all of degree 1), subject to the relations:

$$\begin{array}{ll}
(R^1_{ij}) & x_{ij}^2 = 0 \;\; \text{for all } i < j \\
(R^2_{ij,kl}) & x_{ij}x_{kl} - x_{kl}x_{ij} = 0 \;\; \text{for all } i < j,\; k < l \\
(R^3_{ijk}) & x_{ij}x_{jk} - x_{jk}x_{ik} - x_{ik}x_{ij} = 0 \;\; \text{for all } i < j < k \\
(R^4_{ijk}) & x_{jk}x_{ij} - x_{ik}x_{jk} - x_{ij}x_{ik} = 0 \;\; \text{for all } i < j < k.
\end{array}$$

Despite the deep and multidisciplinary study of these algebras, several fundamental problems regarding their structure remain open. One of the most intriguing problems is: which Fomin-Kirillov algebra are finite-dimensional? The answer to this problem is known only for $n \leq 5$, whose (finite) dimensions and Hilbert series have been computed [2, 14].

Generalized Fomin-Kirillov Algebras

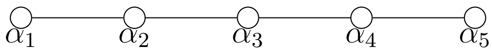
The **generalized Fomin-Kirillov (FK) algebra** extends the classical FK algebra $\mathcal{FK}_n$ by associating it with a subgraph G of the complete graph K_n. These algebras, denoted $\mathcal{FK}_G$, are subalgebras of $\mathcal{FK}_n$ and are generated by the edge variables corresponding to the edges of G. Their structure captures both combinatorial and algebraic properties, often reflecting those of Coxeter and nil-Coxeter algebras.

Let K_n be the **complete graph** on n vertices $\{1, 2, \ldots, n\}$, with its edges identified as unordered pairs $\{i, j\}$. The **generalized Fomin-Kirillov algebra** associated with a subgraph $G \subseteq K_n$ with edge set E is defined as the subalgebra of $\mathcal{FK}_n$ generated by the elements x_{ij} corresponding to the edges of G:

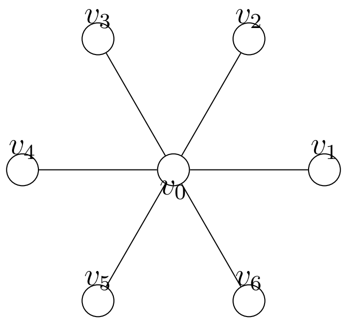
$$\mathcal{FK}_G = \langle x_{ij} \mid \{i, j\} \in E \rangle \subseteq \mathcal{FK}_n.$$

Since the full Fomin-Kirillov algebra $\mathcal{FK}_n$ corresponds to the complete graph K_n, we have $\mathcal{FK}_{K_n} = \mathcal{FK}_n$.

Example 11. Dynkin diagram for type A_n (the example below is for $n = 5$



Example 12. Star $K_{1,n}$ (the example below is for $n = 6$)



3.2 Truncated point schemes of $\mathcal{FK}_n$

Theorem 1. Let $n \geq 3$ be an integer. Then $\mathcal{FK}_n$ admits no truncated point modules of degree greater than 1.

For a graded algebra A one can measure the optimal degree of truncated point modules of A by $p(A) := \sup\{i \in \mathbb{N} \mid \mathcal{P}_i(A) \neq \emptyset\}$. Thus, $p(\mathcal{FK}_n) = 1$, the smallest possible value[1].

In the space of 2-truncated point modules:

$$\mathcal{P}_2(\mathcal{FK}_n) \subseteq \mathbb{P}^{\binom{n}{2}-1} \times \mathbb{P}^{\binom{n}{2}-1}$$

with homogeneous coordinates $([x_{12} : \cdots : x_{n-1\ n}], [y_{12} : \cdots : y_{n-1\ n}])$, we have the following polynomial equations:

$$\begin{array}{ll}
(E^1_{ij}) & y_{ij}x_{ij} = 0 \text{ for all } i < j \\
(E^2_{ij,kl}) & y_{ij}x_{kl} - y_{kl}x_{ij} = 0 \text{ for all } i < j,\ k < l \\
(E^3_{ijk}) & y_{ij}x_{jk} - y_{jk}x_{ik} - y_{ik}x_{ij} = 0 \text{ for all } i < j < k \\
(E^4_{ijk}) & y_{jk}x_{ij} - y_{ik}x_{jk} - y_{ij}x_{ik} = 0 \text{ for all } i < j < k.
\end{array}$$

(They correspond from relations defined in section 3.1)

[1]Notice that $\mathcal{P}_1(A) \cong \mathbb{P}(A_1)$.

Proposition 1. The algebra $\mathcal{FK}_3$ admits no truncated point modules of degree greater than 1.

Proof. Consider:

$$\mathcal{P}_2(\mathcal{FK}_3) \subseteq \mathbb{P}^2_{x_{12},x_{13},x_{23}} \times \mathbb{P}^2_{y_{12},y_{13},y_{23}}$$

and organize the defining polynomial equations in a matrix form:

$$\begin{matrix} (E^3_{123}) \\ (E^4_{123}) \\ (E^1_{12}) \\ (E^1_{13}) \\ (E^1_{23}) \end{matrix} \quad \underbrace{\begin{pmatrix} x_{23} & -x_{12} & -x_{13} \\ -x_{13} & -x_{23} & x_{12} \\ x_{12} & 0 & 0 \\ 0 & x_{13} & 0 \\ 0 & 0 & x_{23} \end{pmatrix}}_{M} \begin{pmatrix} y_{12} \\ y_{13} \\ y_{23} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

If x_{12}, x_{13}, x_{23} are all non-zero then $M(3,4,5)$ is invertible. If exactly one of x_{12}, x_{13}, x_{23} is non-zero then one of $M(1,2,3), M(1,2,4), M(1,2,5)$ is invertible (respectively). In any case, we see that $\mathrm{rank}(M) = 3$, so the only solution to the homogeneous equation $M \cdot \vec{y} = \vec{0}$ is the trivial solution, hence $y_{12} = y_{13} = y_{23} = 0$. This shows that $\mathcal{P}_2(\mathcal{FK}_3) = \emptyset$. □

Although we can see by considering the subalgebra $A \subseteq \mathcal{FK}_n$ generated by $\{x_{pq} | p, q \in \{i, j, k\}\}$. Notice that $A \cong \mathcal{FK}_3$. Indeed, if at least one of r, s, t, u is not in $\{i, j, k\}$ then the relations $(R^1_{rs}), (R^2_{rs,tu}), (R^3_{rst}), (R^4_{rst})$ all become trivial modulo the ideal generated by $\{x_{pq} | p \notin \{i, j, k\}$ or $q \in \{i, j, k\}\}$. By restriction, M is also a graded A-module, and by the way we picked x_{ij}, x_{kl}, M is generated by e_0 (as an A-module). This is a contadiction.

Now for $\mathcal{FK}_4$:

Proposition 2. The algebra $\mathcal{FK}_4$ admits no truncated point modules of degree greater than 1.

Proof. Consider:

$$\mathcal{P}_2(\mathcal{FK}_4) \subseteq \mathbb{P}^5_{x_{12},x_{13},x_{14},x_{23},x_{24},x_{34}} \times \mathbb{P}^5_{y_{12},y_{13},y_{14},y_{23},y_{24},y_{34}}$$

and assume that there is some point $\mathbf{x} = ([x_{12} \colon \cdots : x_{34}], [y_{12} \colon \cdots : y_{34}])$ in $\mathcal{P}_2(\mathcal{FK}_4)$. We may assume that $x_{12} \neq 0$, since at least one of the coordinates x_{ij} is non-zero and $\mathrm{Aut}(\mathcal{FK}_n)$ acts transitively on the generators. Furthermore, we may assume that $x_{12} = 1$. Now by (E^1_{12}), we have that $y_{12} = 0$. By $(E^2_{12,34})$, we have $y_{12}x_{34} = y_{34}x_{12}$ so $y_{34} = 0$. The following homogeneous system holds:

$$\begin{matrix}(E^3_{123})\\ (E^4_{123})\\ (E^3_{124})\\ (E^4_{124})\\ (E^2_{13,24})\\ (E^1_{14})\\ (E^1_{23})\\ (E^1_{24})\end{matrix} \quad \underbrace{\begin{pmatrix} 1 & 0 & x_{13} & 0 \\ -x_{23} & 0 & 1 & 0 \\ 0 & 1 & 0 & x_{14} \\ 0 & -x_{24} & 0 & 1 \\ x_{24} & 0 & 0 & -x_{13} \\ 0 & x_{14} & 0 & 0 \\ 0 & 0 & x_{23} & 0 \\ 0 & 0 & 0 & x_{24} \end{pmatrix}}_{M} \begin{pmatrix} y_{13} \\ y_{14} \\ y_{23} \\ y_{24} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now M is row equivalent (by adding a multiple of the 1st row to the 2nd row and a multiple of the 3rd row to the 4th row) to:

$$M' = \begin{pmatrix} 1 & 0 & x_{13} & 0 \\ 0 & 0 & 1 + x_{13}x_{23} & 0 \\ 0 & 1 & 0 & x_{14} \\ 0 & 0 & 0 & 1 + x_{14}x_{24} \\ x_{24} & 0 & 0 & -x_{13} \\ 0 & x_{14} & 0 & 0 \\ 0 & 0 & x_{23} & 0 \\ 0 & 0 & 0 & x_{24} \end{pmatrix}.$$

If both $1 + x_{13}x_{23}, 1 + x_{14}x_{24}$ are non-zero then $M'(1, 2, 3, 4)$ is invertible. Assume otherwise. If $1 + x_{13}x_{23} = 0$ then in particular $x_{13}, x_{23} \neq 0$, and $M'(1, 3, 5, 7)$ is invertible. If $1 + x_{13}x_{23} \neq 0$ but $1 + x_{14}x_{24} = 0$ then in particular $x_{14}, x_{24} \neq 0$, and $M'(1, 2, 3, 8)$ is invertible. It follows that in any case, $\operatorname{rank}(M) = \operatorname{rank}(M') = 4$ and therefore the only solution to the homogeneous equation $M \cdot \vec{y} = \vec{0}$ is when $\vec{y} = 0$, so $y_{12} = y_{13} = y_{14} = y_{23} = y_{24} = y_{34} = 0$. It follows that $\mathcal{P}_2(\mathcal{FK}_4) = \emptyset$. □

Proof of Theorem 1. It suffices to prove the claim for truncated point modules of degree 2, since we have a sequence of morphisms given by truncation:

$$\cdots \to \mathcal{P}_3(\mathcal{FK}_n) \to \mathcal{P}_2(\mathcal{FK}_n).$$

Let $n \geq 3$ and consider a truncated point module M over $\mathcal{E}_n$ of degree 2 and fix a homogeneous basis, say, $M = ke_0 \oplus ke_1 \oplus ke_2$. Pick arbitrary distinct generators $x_{ij}, x_{kl} \in \mathcal{FK}_n$ (the indices i, j, k, l need not be distinct). Consider the subalgebra $A \subseteq \mathcal{FK}_n$ generated by $\{x_{pq} | p, q \in \{i, j, k, l\}\}$. Notice that $A \cong \mathcal{FK}_3$ (if $|\{i, j, k, l\}| = 3$) or $A \cong \mathcal{FK}_4$ (if $|\{i, j, k, l\}| = 4$) as graded algebras. Indeed, if at least one of r, s, t, u is not in $\{i, j, k, l\}$ then the relations $(R^1_{rs}), (R^2_{rs,tu}), (R^3_{rst}), (R^4_{rst})$ all become trivial modulo the ideal generated by $\{x_{pq} | p \notin \{i, j, k, l\}$ or $q \in \{i, j, k, l\}\}$.

By restriction, M is also a graded A-module. By Propositions 1,2, M cannot be generated by e_0. Therefore, either $A_1 \cdot e_0 = 0$ or $A_1 \cdot e_1 = 0$, so, since A is generated in degree 1, $A_2 \cdot e_0 = (A_1)^2 \cdot e_0 = 0$. In particular, $x_{ij}x_{kl} \cdot e_0 = 0$. Since x_{ij}, x_{kl} were arbitrary (distinct) generators of $\mathcal{FK}_n$ (and all of the generators are squared zero), it follows that $(\mathcal{FK}_n)_2 \cdot e_0 = 0$, so M is not generated by its degree-0 component. Therefore $\mathcal{FK}_n$ admits no truncated point modules of degree ≥ 2. □

3.3 Truncated point schemes of generalized Fomin-Kirillov algebras

Remark. If we have an algebra A and subalgebra B, then if we have a module over A and if M is a module over A then M is a module over B. But if M is a point module over A then it always a point module over B.

Lemma 1. Suppose that G is a finite connected graph that is not a complete graph, then there exist vertices p, q and r such that pq, qr is an edge but pr is not an edge.

Proof. The lemma is the same as: If G is a finite connected graph such that for every three vertices p, q, r, the existence of edges pq and qr implies the existence of edge pr, then G is a complete graph. We prove this using induction on the number of edges in a path between any two vertices p and r.
Base Case:
If the path between p and r consists of only one edge ($n = 1$), then p and r are directly connected. This satisfies the definition of a complete graph for this case.
Inductive Hypothesis:
Assume that for any pair of vertices p, r connected by a path of n edges, the given property ensures that p and r are directly connected by an edge.
Inductive Step:
We need to show that if p and r are connected by a path of $n + 1$ edges, then they must be directly connected.

- Suppose p and r are connected by a path of $n + 1$ edges.
- This means there exists an intermediate sequence of vertices $p = q_0, q_1, q_2, \ldots, q_k = r$ forming a path.
- From the inductive hypothesis, p is directly connected to q_k by an edge, and q_k is also directly connected to r by an edge.
- By the assumption given in the lemma, for any three consecutive vertices q_{i-1}, q_i, q_{i+1} in this path, if edges $q_{i-1}q_i$ and q_iq_{i+1} exist, then the edge $q_{i-1}q_{i+1}$ must also exist.
- Applying this property iteratively along the path, we conclude that an edge must exist between p and r.
- Hence, p and r are directly connected.

By induction, every pair of vertices in G is directly connected by an edge, meaning that G is complete. Since every pair of vertices in G is directly connected by an edge, the graph satisfies the definition of a complete graph. Thus, the lemma is proved.

□

Lemma 2. $\mathcal{FK}_{A_3}$ has a truncated point module of degree 2.

Proof. To establish the existence of a truncated point module of degree 2, we proceed as follows:

The algebra $\mathcal{FK}_{A_3}$ is generated by two elements a and b with relations:

$$a^2 = 0, \quad b^2 = 0, \quad aba = bab.$$

A spanning set of $\mathcal{FK}_{A_3}$ consists of the following monomials:

$$\{1, a, b, ab, ba, aba\}.$$

The left ideal $L = \langle a \rangle$ consists of all elements of the form $x \cdot a$, where $x \in \mathcal{FK}_{A_3}$. Using the basis elements, we compute:

$$1 \cdot a = a, \quad a \cdot a = 0, \quad b \cdot a = ba, \quad ab \cdot a = aba, \quad ba \cdot a = ba^2 = 0, \quad aba \cdot a = 0.$$

Thus, the basis of $L = \langle a \rangle$ is:

$$\{a, ba, aba\}.$$

The quotient module $\mathcal{FK}_{A_3}/L$ has a basis consisting of (the cosets of) the basis elements of $\mathcal{FK}_{A_3}$ that are not in L:

$$\{1, b, ab\}.$$

Therefore, $\mathcal{FK}_{A_3}/L = k \oplus kb \oplus kab$ is a truncated point module of degree 2, the proof is complete. □

Proposition 3. Let G be a graph on n vertices. Then $\mathcal{P}_2(\mathcal{FK}_G) = \emptyset$ if and only if G is a disjoint union of complete graphs.

Proof. Let $G = \bigsqcup_{i=1}^{m} K_{n_i}$ be a disjoint union of complete graphs. Notice that any generator of $\mathcal{FK}_G$ commutes with any other generator, as long as they correspond to edges from separate connected components. If there is a truncated point module $M = ke_0 \oplus ke_1 \oplus ke_2$ over $\mathcal{FK}_G$ — and hence over $\bigotimes_{i=1}^{m} \mathcal{FK}_{n_i}$ — then for some $1 \leq i, j \leq m$, we have $x_{pq} \in \mathcal{FK}_{K_{n_i}}$ and $x_{rs} \in \mathcal{FK}_{K_{n_j}}$ such that $x_{pq} \cdot e_0 \neq 0, x_{rs} \cdot e_1 \neq 0$. If $i = j$, we obtain a contradiction to Theorem 1, since then M becomes a degree-2 point module over $\mathcal{FK}_{n_i}$. The edges x_{pq} and x_{rs} commute and now, if $i \neq j$ then

$$0 \neq c_1 c_2 e_2 = x_{rs} \cdot c_1 e_1 = x_{rs} x_{pq} \cdot e_0 = x_{pq} \cdot (x_{rs} \cdot e_0),$$

but we claim that a contradiction to the above statement. To see why this is true let $x_{rs} \cdot e_0 = c' e_1$. Now $0 = x_{rs}^2 \cdot e_0 = x_{rs} \cdot c' e_1 = c' c_2 e_2$ Since $c_2 \neq 0$, then $c' = 0$, so therefore $x_{rs} \cdot e_0 = 0$, which is a contradiction.

Conversely, assume that G is not a disjoint union of complete graphs; pick a connected component that is not a complete graph. From lemma 1 we can find distinct vertices p, q, r such that the edges $\{p, q\}, \{q, r\}$ are in G but $\{p, r\}$ is not. Let K be the complete graph on p, q, r, and observe that $\mathcal{FK}_3 \cong \mathcal{FK}_K \cong k\langle x_{pq}, x_{pr}, x_{qr} \rangle \subseteq \mathcal{FK}_n$. Thus the subgraph of G consisting of the vertices p, q, r is A_3 (this is a graph with three vertices and two edges) and from Lemma 2 we oberve that $\mathcal{FK}_{A_3}/L \cong k \oplus kb \oplus kab$ is a truncated point module of degree 2. Notice that $\mathcal{FK}_G \twoheadrightarrow \mathcal{FK}_{A_3}$, moding out by all edges incidenting with a vertex outside $\{p, q, r\}$. By inflation, we obtain a 2-truncated point module over $\mathcal{FK}_G$, and it follows that $\mathcal{P}_2(\mathcal{FK}_G) \neq \emptyset$. □

Proposition 4. Let $n \geq 3$ and let A_n be the graph with vertices $\{1, \dots, n\}$ and with edges $\{i, i+1\}$ for $1 \leq i \leq n-1$. Then:

$$\#\mathcal{P}_d(\mathcal{FK}_{A_n}) = 0 \text{ if } d \geq n$$

Proof. Let

$$(x^0, \dots, x^{n-1}) = ([x_1^0 : \cdots : x_{n-1}^0], \dots, [x_1^{n-1} : \cdots : x_{n-1}^{n-1}]) \in \mathcal{P}_n(\mathcal{FK}_{A_n}) \subseteq \left(\mathbb{P}^{n-2}\right)^{\times n}$$

be a point in projective coordinates representing a truncated point module of $\mathcal{FK}_{A_n}$. In other words, enumerate the edges in A_n as $x_1, \dots, x_{n-1}$, which we identify with the generators of $\mathcal{FK}_{A_n}$, then $x_i \cdot e_r = x_i^r \cdot e_{r+1}$, where $ke_0 \oplus \cdots \oplus ke_n$ is a point module over $\mathcal{FK}_{A_n}$. We begin by applying the relations in $\mathcal{FK}_{A_n}$, given by [4, Theorem 6.1], to equations on the x_i^r's. Indeed, $\mathcal{FK}_{A_n}$ has the following presentation:

$$k\langle x_1, \dots, x_{n-1}\rangle / \left\langle x_i^2, x_ix_j - x_jx_i, x_kx_{k+1}x_k - x_{k+1}x_kx_{k+1} \ : \ |i-j| > 1, \ 1 \leq k \leq n-2 \right\rangle$$

From which we conclude the following equations on the coordinates of $\mathcal{P}_d(\mathcal{FK}_{A_n})$

Relation in $\mathcal{FK}_{A_n}$	Representation in $\mathcal{P}_d(\mathcal{FK}_{A_n})$
(A_i^r) $\quad x_i^2 = 0$	$x_i^{r+1}x_i^r = 0$
(B_{ij}^r) $\quad x_ix_j = x_jx_i$ $\|i-j\| > 1$	$x_i^{r+1}x_j^r = x_j^{r+1}x_i^r$
(C_k^s) $\quad x_kx_kx_{k+1} = x_kx_{k+1}x_kx_{k+1}$	$x_k^{s+2}x_k^{s+1}x_k^s = x_k^{s+2}x_{k+1}^{s+1}x_k^s$

Let us use the above equations on the points $x^0, \dots, x^{n-1}$ in the $(n-2)$-dimensional projective space, to show after n steps all entries of the final point would be 0, resulting in a contradiction. First, we examine the effect of each one of the above relations on the combinatorial structure of these points.

Applying (A_i^r): Suppose

$$x^r : \left[\cdots \quad x_i^r \quad \cdots\right], \quad x_i^r \neq 0$$

(since at least one entry for a vector in projective variety must be $\neq 0$) and let

$$x^{r+1} : \left[\cdots \quad x_i^{r+1} \quad \cdots\right].$$

The relation A_i^r implies that if $x_i^r \neq 0$, then $x_i^{r+1} = 0$.

Applying (B_{ij}^r): Here, j is such that $|i-j| > 1$, meaning x_i and x_j commute in $\mathcal{FK}_{A_n}$. We consider the coordinates of x^r, x^{r+1}:

$$\begin{array}{ccccccccc} x^r & = & [x_1^r & \cdots & x_i^r & \cdots & x_j^r & \cdots & x_n^r] \\ x^{r+1} & = & [x_1^{r+1} & \cdots & x_i^{r+1} & \cdots & x_j^{r+1} & \cdots & x_n^{r+1}]. \end{array}$$

Assuming $x_i^r \neq 0$, the relation (A_i^r) gives $x_i^{r+1} = 0$ as we stablished above. Then, by (B_{ij}^r),

$$x_i^{r+1}x_j^r = x_j^{r+1}x_i^r$$

and since $x_i^{r+1} = 0$ and $x_i^r \neq 0$, it follows that $x_j^{r+1} = 0$.

Applying (C_k^r): We examine the coordinates involving x_i and x_{i+1}:

$$\begin{array}{r}x^r : [x_1^r \quad \cdots \quad x_i^r \quad x_{i+1}^r \quad \cdots \quad x_n^r] \\ x^{r+1} : [x_1^{r+1} \quad \cdots \quad x_i^{r+1} \quad x_{i+1}^{r+1} \quad \cdots \quad x_n^{r+1}] \\ x^{r+2} : [x_1^{r+2} \quad \cdots \quad x_i^{r+2} \quad x_{i+1}^{r+2} \quad \cdots \quad x_n^{r+2}]\end{array}$$

Suppose $x_i^r \neq 0$. Then (A_i^r) gives $x_i^{r+1} = 0$. Now suppose $x_{i+1}^{r+1} \neq 0$. Then (A_i^r) gives $x_{i+1}^{r+2} = 0$. From (C_k^r), we know:

$$x_i^r x_{i+1}^{r+1} x_i^{r+2} = x_{i+1}^r x_i^{r+1} x_{i+1}^{r+2}.$$

From our assumptions, the right-hand side is zero since $x_i^{r+1} = 0$. Therefore, the left-hand side is also zero:

$$x_i^r x_{i+1}^{r+1} x_i^{r+2} = 0 \Rightarrow x_i^{r+2} = 0,$$

since $x_i^r, x_i^{r+1} \neq 0$.

Using the above relations to compute $\mathcal{P}_n(\mathcal{FK}_{A_n})$:

$$\begin{array}{r}x^0 = [\quad \cdots \quad x_{i-1}^0 \quad x_i^0 \quad x_{i+1}^0 \quad \cdots] \\ x^1 = [\quad \cdots \quad x_{i-1}^1 \quad x_i^1 \quad x_{i+1}^1 \quad \cdots]\end{array}$$

Assume $x_i^0 \neq 0$, then A_i^r gives $x_i^1 = 0$. Also, B_{ij}^r forces $x_j^1 = 0$ for all $|j - i| > 1$. So we have:

$$x^1 : \big[0 \cdots \quad 0 \quad 0 \quad x_{i-1}^1 \quad 0 \quad x_{i+1}^1 \quad 0 \quad 0 \quad \cdots\big]$$

Since x^1 is a point in a projective space, at least one of x_{i-1}^1 or x_{i+1}^1 must be non-zero. Consider

$$x^2 = [\quad \cdots \quad x_{i-2}^2 \quad x_{i-1}^2 \quad x_i^2 \quad x_{i+1}^2 \quad x_{i+2}^2 \quad \cdots].$$

Suppose $x_{i-1}^1 \neq 0$ and $x_{i+1}^1 = 0$. Then:

- A_i^r gives $x_{i-1}^2 = 0$
- C_k^r gives $x_i^2 = 0$ (since $x_i^0 x_{i-1}^1 x_i^2 = 0 \Rightarrow x_i^2 = 0$)
- B_{ij}^r gives $x_j^2 = 0$ for all $|j - i| > 1$

Thus:

$$x^2 = [0 \quad \cdots \quad 0 \quad x_{i-2}^2 \quad 0 \quad 0 \quad 0 \quad \cdots \quad 0]$$

Continuing in this manner inductively:

$$\begin{array}{rcl}x^3 & = & [0 \quad \cdots \quad 0 \quad x_{i-3}^3 \quad 0 \quad 0 \quad \cdots \quad 0] \\ & \vdots & \\ x^{n-1} & = & [x_{i-n+1}^{n-1} \quad 0 \quad 0 \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad 0] \\ x^n & = & [0 \quad 0 \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad 0]\end{array}$$

We observe that the non-zero component shifts one position to the left at each degree, but since there are only $n-1$ positions, we cannot shift n times, as indicated in the last line above. Therefore, all entries of x^n must be zero, implying $\mathcal{P}_n(\mathcal{FK}_{A_n}) = \varnothing$.

If, instead, we assume $x^1_{i+1} \neq 0$ $x^1_{i-1} = 0$ and , we would obtain:

$$\begin{array}{rllllllll}
x^2: [0 & \cdots & 0 & 0 & 0 & 0 & x^2_{i+2} & \cdots & 0] \\
x^3: [0 & \cdots & 0 & 0 & 0 & 0 & x^3_{i+3} & \cdots & 0] \\
 & \vdots & & & & & & & \\
x^{n-1}: [0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & x^{n-1}_{i+n-1}] \\
x^n: [0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \cdots & 0]
\end{array}$$

Similarly here we observe that the non-zero component shifts one position to the right at each degree, but since there are only $n-1$ positions, we cannot shift n times. Therefore, all entries of x^n must be zero, implying $\mathcal{P}_n(\mathcal{FK}_{A_n}) = \varnothing$.

Finally, if both $x^1_{i-1} \neq 0$ and $x^1_{i+1} \neq 0$, then:

$$x^1_{i-1} \neq 0 \Rightarrow x^2_j = 0 \text{ for all } j \neq (i-1) \pm 1 \text{ by } (B^r_{ij})$$
$$x^1_{i+1} \neq 0 \Rightarrow x^2_j = 0 \text{ for all } j \neq (i+1) \pm 1 \text{ by } (B^r_{ij})$$

so $x^2 = [0 \cdots 0]$, a contradiction, so all entries of x^2 are zero, contradicting the property for nonzero vectors in a projective representation. Hence, $\mathcal{P}_n(\mathcal{FK}_{A_n}) = \varnothing$, and therefore $\#\mathcal{P}_d(\mathcal{FK}_{A_n}) = 0$ for all $d \geq n$. □

Remark. In fact, the above proof can be used to compute $\mathcal{P}_d(\mathcal{FK}_{A_n})$ for $d < n$, which are always finite sets. This was done in our paper [1, Proposition 3.2].

In [2, Lemma 7.2] (see also [4]), the following 'cyclic relation' in $\mathcal{FK}_n$ was computed: for every $m = 3, \ldots, n$ and distinct vertices $1 \leq a_1, \ldots, a_m \leq n$ in the complete graph K_n:

$$\sum_{i=2}^{m} x_{a_1 a_i} x_{a_1 a_{i+1}} \cdots x_{a_1 a_m} x_{a_1 a_2} \cdots x_{a_1 a_i} = 0. \tag{1}$$

(Notice that if $b < a$ then x_{ab} in (1) is equal to $-x_{ba}$; see [2].)

Proposition 5. Let $K_{1,n}$ be the star graph with $n+1$ vertices and n edges. Then $\mathcal{P}_d(\mathcal{FK}_{K_{1,n}}) = \emptyset$ for $d \geq n+1$.

Proof. For simplicity, let us label the vertex at the center of the star by 0 and the rest of the vertices by $1, \ldots, n$. Denote the n generators of $\mathcal{FK}_{K_{1,n}}$ (corresponding to the n edges of the star) by $x_1, \ldots, x_n$, so $x_i = x_{0i}$ is the edge connecting the vertices 0 and i. We argue that $\mathcal{P}_{n+1}(\mathcal{FK}_{K_{1,n}}) = \emptyset$. Otherwise, let:

$$\mathbf{x} = (x^0, \ldots, x^n) \in \mathcal{P}_{n+1}(\mathcal{FK}_{K_{1,n}}) \subseteq \left(\mathbb{P}^{n-1}\right)^{\times(n+1)}$$

where $x^i = [x^i_1 : \cdots : x^i_n]$. For each $0 \leq i \leq n$, let $S_i \subseteq \{1, \ldots, n\}$ be the support of x^i, namely,

$$\text{supp}_i = \{1 \leq j \leq n \mid x^i_j \neq 0\}.$$

Notice that $\text{supp}_i \neq \emptyset$ for all $0 \leq i \leq n$. We must have some $0 \leq i < j \leq n$ such that $\text{supp}_i \cap \text{supp}_j \neq \emptyset$. Otherwise, for all i and j, $\text{supp}_i \cap \text{supp}_j = \emptyset$ so $|\text{supp}_0 \cup \cdots \cup \text{supp}_n| = |\text{supp}_0| + \cdots + |\text{supp}_n| \geq n+1$, a contradiction.

Suppose that $\text{supp}_i \cap \text{supp}_j \neq \emptyset$ for some $0 \leq i < j \leq n$ with the smallest possible $j - i$. We first observe that $j > i+1$: let us assume $r \in \text{supp}_i$ so $x_i^r \neq 0$, then using (A_i^r) (from Proposition (4), but which applies in general) we have $x_i^{r+1} = 0$, so $r \notin \text{supp}_{i+1}$. Pick $t_i \in \text{supp}_i$, $t_{i+1} \in \text{supp}_{i+1}, \ldots, t_j = t_i \in \text{supp}_j$ such that $t_i, \ldots, t_{j-1}$ are all distinct, because, assume there were some $i \leq i' < j' \leq j-1$ such that $t_{i'} = t_{j'}$, which would mean that $\text{supp}_{i'} \cap \text{supp}_{j'} \neq \emptyset$, and $j' - i' < j - i$ contradicting the minimality of $j - i$ in the way we picked i, j.

By Equation (1) for $a_1 = 0, a_2 = t_i, a_3 = t_{j-1}, \ldots, a_{j-i+1} = t_{i+1}$

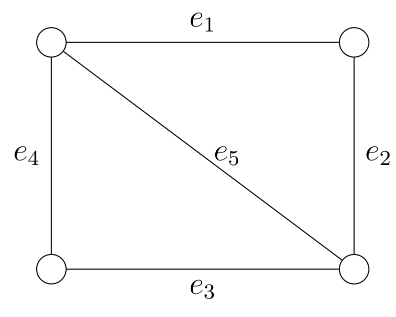
$$\begin{aligned} x^j_{t_i} x^{j-1}_{t_{j-1}} \cdots x^{i+1}_{t_{i+1}} x^i_{t_i} &+ x^j_{t_{j-1}} x^{j-1}_{t_{j-2}} \cdots x^{i+1}_{t_i} x^i_{t_{j-1}} + \cdots \\ &+ x^j_{t_{i+1}} x^{j-1}_{t_i} \cdots x^{i+1}_{t_{i+2}} x^i_{t_{i+1}} = 0. \end{aligned} \tag{2}$$

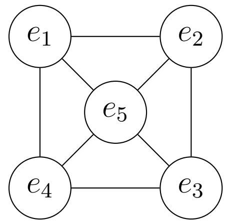
Notice that in (2), all of the monomials except for the first one vanish, since $t_{i+1}, \ldots, t_{j-1} \notin \text{supp}_i$; but the first monomial is non-zero, since $t_{i+l} \in \text{supp}_{i+l}$ for $l = 0, \ldots, j-i$ (recall that $t_i = t_j$), a contradiction. It follows that all of the supports are pairwise disjoint. Since these supports must be non-empty, it follows that for $d \geq n+1$ we have at least $n+1$ pairwise disjoint non-empty subsets of $\{1, \ldots, n\}$, a contradiction; hence $\mathcal{P}_{n+1}(\mathcal{FK}_{K_{1,n}}) = \emptyset$. □

Line Graph of a Graph

Let G be a graph with a set of vertices and edges. The **line graph** of G, denoted as $L(G)$, is defined as follows: - The vertices of $L(G)$ correspond to the edges of G. - Two vertices in $L(G)$ are adjacent if and only if their corresponding edges in G share a common vertex.

Example: Graph and Its Line Graph





Line Graph $L(G)$

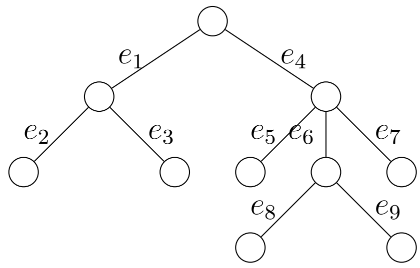
Bi-Connected Component

A **bi-connected component** of a graph is a maximal subgraph where the removal of any single vertex does not disconnect the subgraph. These are crucial in understanding the articulation points and connectivity of graphs.

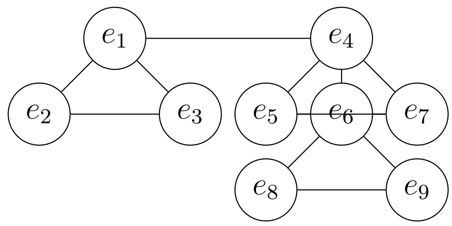
Block Graph

A **block graph** is a graph in which every bi-connected component (block) is a complete graph. The line graph of a tree is always a block graph.

Example: Tree and Its Line Graph



Tree T



Line Graph $L(T)$

According to [6, Theorem 8.5] every tree $\mathcal{T}$ is a block graph.

Theorem 2. For every tree $\mathcal{T}$ we have that $p(\mathcal{FK}_{\mathcal{T}})$ is at most the number of edges of $\mathcal{T}$, and this bound is best possible.

Proof. Let $\mathcal{T}$ be a tree with n edges, say, $x_1, \dots, x_n$. By abuse of notation, let us identify $x_1, \dots, x_n$ with the corresponding generators of $\mathcal{FK}_{\mathcal{T}}$. Assume to the contrary that $\mathcal{FK}_{\mathcal{T}}$ admits a truncated point module:

$$M = ke_0 \oplus \cdots \oplus ke_{n+1}$$

and pick, for each $0 \le i \le n$, an edge x_{a_i} such that $x_{a_i} \cdot e_i \ne 0$ (namely, $x_{a_i} \cdot e_i$ is a non-zero scalar multiple of e_{i+1}). Thus $a_0, \dots, a_n \in \{1, \dots, n\}$. Furthermore, we claim that for each $0 \le i \le n-1$, the edges $x_{a_i}, x_{a_{i+1}}$ share a common vertex. For otherwise, their corresponding generators would commute with each other by the defining relations of the Fomin-Kirillov algebra, so say $x_{a_i} \cdot e_i = \lambda e_{i+1}$ and $x_{a_{i+1}} \cdot e_{i+1} = \lambda' \cdot e_{i+2}$ for $\lambda, \lambda' \ne 0$ and $x_{a_i} x_{a_{i+1}} \cdot e_i = x_{a_{i+1}} x_{a_i} \cdot e_i = \lambda' \lambda e_{i+2} \ne 0$, in particular $x_{a_{i+1}} \cdot e_i \ne 0$ (because if $x_{a_{i+1}} \cdot e_i = 0$ then $x_{a_i} x_{a_{i+1}} \cdot e_i = 0$ but $x_{a_i} x_{a_{i+1}} \cdot e_i = \lambda' \lambda e_{i+2}$) so $x_{a_{i+1}} \cdot e_i = \mu \cdot e_{i+1}$ for some $\mu \ne 0$ and $x_{a_{i+1}}^2 \cdot e_i = \lambda' \mu e_{i+2} \ne 0$, but $x_{a_{i+1}}^2 = 0$ (as per the defining relations fo the Fomin-Kirillov algebra).

By the Pigeonhole Principle, there exist $0 \le i < j \le n$ such that $a_i = a_j$; we may further assume that $|i - j|$ is as small as possible. (Notice that $j \ge i + 2$, since otherwise $x_{a_{i+1}} x_{a_i} \cdot e_i = x_{a_i}^2 \cdot e_i = 0$.) Hence the sequence of edges:

$$C \colon x_{a_i}, x_{a_{i+1}}, \dots, x_{a_{j-1}}, x_{a_j} = x_{a_i}$$

is a cycle (hence a bi-connected subgraph) in the line graph $L(\mathcal{T})$. Fix a maximal bi-connected component containing C; since $L(\mathcal{T})$ is a block graph (see [6, Theorem 8.5] for more information on line graphs), each bi-connected component is a clique. In particular, each two edges among $x_{a_i}, \dots, x_{a_j}$ share a common vertex in $\mathcal{T}$.

For each $i \le t \le j - 1$, let v_t denote the (unique) common vertex incidenting both x_{a_t} and $x_{a_{t+1}}$. Suppose that for some $i \le t \le j - 2$ we have $v_t \ne v_{t+1}$. Then

$x_{a_{t+1}}$ incidents both v_t, v_{t+1}, so $x_{a_t} \neq x_{a_{t+2}}$ (as if $x_{a_t} = x_{a_{t+2}}$ then x_{a_t} incidents the same vertices as $x_{a_{t+1}}$, so they must be equal to each other, a contradiction to $x_{a_t}^2 = 0$). Since each two edges among $x_{a_i}, \dots, x_{a_j}$ share a common vertex, it follows that $x_{a_t}, x_{a_{t+1}}, x_{a_{t+2}}$ form a triangle in $\mathcal{T}$. This is because we have v_t that is the common vertex between x_{a_t} and $x_{a_{t+1}}$, we have v_{t+1} that is the common vertex between $x_{a_{t+1}}$ and $x_{a_{t+2}}$. Now if $v_{t+2} = v_{t+1}$ then $x_{a_t} = x_{a_{t+1}}$, similarly if $v_{t+2} = v_t$ then $x_{a_{t+2}} = x_{a_{t+1}}$ and both are not possible and hence v_{t+2} connects x_{a_t} and $x_{a_{t+2}}$ which forms a triangle in $\mathcal{T}$, contradicting that $\mathcal{T}$ is a tree; hence $v_i = \cdots = v_{j-1}$.

In conclusion, $x_{a_i}, \dots, x_{a_j}$ form a star graph $K_{1,j-i}$. Therefore, if we re-name the homogeneous basis elements $e_i, \dots, e_{j+1}$ of M by $f_0, \dots, f_{(j-i)+1}$, respectively, and re-grade them by $\deg(f_0) = 0, \dots, \deg(f_{(j-i)+1}) = j - i + 1$ then:

$$M' = kf_0 \oplus \cdots \oplus ke_{(j-i)+1} = ke_i \oplus \cdots \oplus ke_{j+1}$$

is a degree-$(j-i+1)$ truncated point module over $\mathcal{FK}_{K_{1,j-i}}$, a contradiction to Proposition 5. This, together with the observation from Proposition 4 that $p(\mathcal{FK}_{A_n}) = n - 1$ (recall that A_n has $n - 1$ edges), completes the proof. □

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