On Minimal Domains and Quasi-Reinhardt Domains

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Abstract

We give an overview of Bergman theory, including the Bergman space and the Bergman kernel function. We state and prove some properties of minimal domains and introduce the classification of minimal domains in $\mathbb{C}$. We then present some results from representation theory of compact Lie groups to give a modified definition of quasi-Reinhardt domains, as well as an intrinsic definition. Finally, we use the intrinsic definition to develop some properties of quasi-Reinhardt domains and mappings on them.

Contents

Introduction

Let $D \subset \mathbb{C}^n$ be a domain, and denote the Bergman space of D by $A^2(D) = \mathcal{O}(D) \cap L^2(D)$. The Bergman space is a separable Hilbert space with the inner product inherited from $L^2(D)$ (with respect to the Lebesgue $\mathbb{R}^{2n}$ measure). Let $\{\varphi_j\}_{j=1}^{\infty}$ be a complete orthonormal basis for $A^2(D)$, and one definition of the Bergman kernel function is given by

$$K(z,w) = \sum_{j=1}^{\infty} \varphi_j(z)\overline{\varphi_j(w)}.$$

If D is a bounded domain, then the constant function $c_0 \equiv \mu(D)^{-1/2}$ is in $A^2(D)$ and has norm equal to 1. Therefore, if $\{\varphi_j\}_{j=1}^{\infty}$ is an complete orthonormal basis containing c_0, then

$$K(z,z) = \sum_{j=1}^{\infty} |\varphi(z)|^2 \geq \frac{1}{\mu(D)}.$$

If the equality is achieved at $z_0 \in D$, then D is called a minimal domain with center z_0. In 2021, Robert Xin Dong and John Treuer ([DT21]) compeletely classified minimal domains in $\mathbb{C}$, claiming that they are disks up to subtracting a polar set. However, this result does not generalize to $\mathbb{C}^n$ with $n > 1$. By considering Cartan egg-domains, one can see that there are many minimal domains that are not biholomorphic to both of the unit ball $\mathbb{B}^n$ and the polydisk $\mathbb{D}^n$. Therefore, we want to study a larger class of domain which contains the collection of minimal domains.

Quasi-Reinhardt domains become the central topic of this thesis because every quasi-Reinhardt domain is a minimal domain. A domain D is quasi-Reinhardt with respect to a torus action $\rho_A : \mathbb{T}^r \to \mathrm{GL}_n(\mathbb{C})$, if D is ρ_A-invariant and $\mathcal{O}(\mathbb{C}^n)^{\rho_A} = \mathbb{C}$. These notions and symbols shall be defined in Section 4 of this thesis. Quasi-Reinhardt domains are first studied by Fusheng Deng, Feng Rong, and Fengbai Li ([DR16], [LR19]). However, an argument in [DR16] has a flaw in it, and the definition of quasi-Reinhardt thus need to be modified to remedy this flaw. In this thesis, we will provide a modified definition of quasi-Reinhardt domains. We will also provide an intrinsic definition of quasi-Reinhardt domains and use it to study some properties of mappings on quasi-Reinhardt domains.

1 Bergman Theory

In this section, we provide an introduction and exposition of Bergman theory, which would be essential for the discussion of minimal domains in Section 2. [Kra13] will be the main reference for this exposition.

1.1 Bergman Space

Let $D \subset \mathbb{C}^n$ be a bounded domain (open and connected), and let $d\mu$ be the $\mathbb{R}^{2n}$ Lebesgue volume measure on D. Define the **Bergman space** by the collection of all L^2 holomorphic functions on D.

$$A^2(D) = \left\{ f : D \to \mathbb{C} \text{ holomorphic} \;\middle|\; \int_D |f(z)|^2 d\mu(z) < \infty \right\}.$$

equipped with the inner product and norm inherited from $L^2(D)$:

$$\langle f, g \rangle = \int_D f(z)\overline{g(z)} d\mu(z)$$

and

$$||f||_{A^2(D)} := \left(\int_D |f(z)|^2 d\mu(z) \right)^{1/2}.$$

That is, $A^2(D) = \mathcal{O}(D) \cap L^2(D)$, where $\mathcal{O}(D)$ is the space of holomorphic functions on D. We have that the uniform norm of $f \in A^2(D)$ on compact subsets of D is bounded by its $A^2(D)$-norm.

Proposition 1.1. ([Kra13], Lemma 1.1.1) Suppose $D \subset \mathbb{C}^n$ is a bounded domain. Let $K \subset D$ be a compact set. Then there is a constant $C_{K,n} > 0$, such that

$$\sup_{z \in K} |f(z)| \leq C_{K,n} ||f||_{A^2(D)}$$

for all $f \in A^2(D)$.

Suppose f_n is a Cauchy sequence in $A^2(D) \subset L^2(D)$. By completeness of L^2, we may find f so that $f_n \to f$ in $L^2(D)$. By Proposition 1.1, we see that this convergence is also uniform over compact subsets of D, and thus f is holomorphic on D. That is, $f \in A^2(D)$, which means we have proved the following:

Proposition 1.2. $A^2(D)$ with the inner product $\langle f, g \rangle = \int_D f(z)\overline{g(z)} d\mu(z)$ is a Hilbert space.

Remark 1.3. As $L^2(D)$ is a separable Hilbert space, so is $A^2(D) \subset L^2(D)$. Therefore, we can find a complete countable orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ for $A^2(D)$.

1.2 Bergman Kernel

We now construct the Bergman Kernel for a given domain $D \subset \mathbb{C}^n$. If $z \in D$, notice that applying Proposition 1.1 with the compact set $K = \{z\}$, we have

$$|f(z)| \leq C ||f||_{A^2(D)}$$

for all $f \in A^2(D)$. Hence, for each fixed $z \in D$, $\Phi_z : A^2(D) \to \mathbb{C}$, the evaluation map $\Phi_z(f) = f(z)$ is a bounded linear functional, and is thus continuous. By Riesz representation theorem, there is some $K_z \in A^2(D)$ with the property that

$$f(z) = \Phi_z(f) = \langle f, K_z \rangle.$$

Definition 1.4. The **Bergman kernel** (on D) is defined by the function $K_D(z, w) = \overline{K_z(w)}$, where $z, w \in D$. When it is clear by the context, we also write $K(z, w)$ in place of $K_D(z, w)$ for the Bergman kernel.

Example 1.5. Let $\mathbb{B}^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_1|^2 + \cdots + |z_n|^2 < 1\}$ be the unit ball in $\mathbb{C}^n$. The Bergman Kernel of the unit ball is given by

$$K_{\mathbb{B}^n}(z, w) = \frac{n!}{\pi^n} \frac{1}{(1 - z \cdot \overline{w})^{n+1}}$$

where $z \cdot \overline{w} = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n$ is the standard inner product on $\mathbb{C}^n$.

We have the following reproducing property of Bergman kernel immediately from the definition.

Proposition 1.6. Let $D \subset \mathbb{C}^2$ be a bounded domain and $K(z, w)$ the associated Bergeman kernel. Then

$$f(z) = \int_D f(w)K(z, w)d\mu(w)$$

for all $z \in D$ and $f \in A^2(D)$.

The Bergman kernel is conjugate symmetric by the reproducing property.

Proposition 1.7. Let D and $K(z, w)$ be the same as in Proposition 1.6. Then $K(z, w) = \overline{K(w, z)}$ for all $z, w \in D$.

Proof. By definition, we have that the function $g_\omega(\zeta) = \overline{K(\omega, \zeta)}$ is in $A^2(D)$ for each fixed $\omega \in D$. Fix $z, w \in D$. Applying the reproducing property yields

$$\overline{K(w, z)} = g_w(z) = \int_D \overline{K(w, t)}K(z, t)d\mu(t) = \overline{\int_D \overline{K(z, t)}K(w, t)d\mu(t)} = \overline{g_z(w)} = K(z, w).$$

□

Proposition 1.8. The Bergman kernel $K(z, w)$ is the unique function $D \times D \to \mathbb{C}$ that is $A^2(D)$ in the z variable, conjugate symmetric (Proposition 1.7), and satisfies the reproducing property (Proposition 1.6).

Proof. Suppose $K'(z, w)$ is another kernel with the given properties. By the same argument as in Proposition 1.7, we obtain

$$K'(z, w) = \overline{K'(w, z)} = \int_D \overline{K'(w, t)}K(z, t)d\mu(t) = \overline{\int_D \overline{K(z, t)}K'(w, t)d\mu(t)} = K(z, w).$$

□

Recall from Remark 1.3 that we may find a complete countable orthonormal basis $\{\varphi_j\}_{j=1}^{\infty}$ for $A^2(D)$. We now give an alternative definition of the Bergman kernel.

Proposition 1.9. (Alternative definition of Bergman kernel) Let $\{\varphi_j\}_{j=1}^{\infty}$ be an orthonormal basis for $A^2(D)$. Consider the series

$$\sum_{j=1}^{\infty} \varphi_j(z)\overline{\varphi_j(w)}.$$

If $S \subset D$ is compact, then the series converges to the Bergman kernel uniformly over $S \times S$.

Remark 1.10. Notice that this definition does not depend on the choice of orthonormal basis. Suppose $D \subset \mathbb{C}^n$ is a bounded domain. Then every constant function is in $A^2(D)$. Therefore, we may use Gram-Schmidt to produce an orthonormal basis that contains the constant function $\varphi_1 \equiv \mu(D)^{-1/2}$, where $\mu(D)$ is the Lebesgue $\mathbb{R}^{2n}$ measure of D. Suppose the orthonormal basis is given by

$$\{\varphi_j\}_{j=1}^{\infty} = \{\mu(D)^{-1/2}\} \cup \{\varphi_j\}_{j=2}^{\infty}.$$

By Proposition 1.9, we have the following diagonal estimate for the Bergman Kernel $K(z,w)$:

$$K(z,z) = \sum_{j=1}^{\infty} \varphi_j(z)\overline{\varphi_j(z)} = \sum_{j=1}^{\infty} |\varphi_j(z)|^2 \geq |\varphi_1(z)|^2 = \frac{1}{\mu(D)}.$$

This leads to one definition of minimal domains, and we will further discuss this in Section 2.

Definition 1.11. Suppose $D \subset \mathbb{C}^n$ is a domain, and $f : D \to \mathbb{C}^n$ is holomorphic. Write $f(z) = (f_1(z), \ldots, f_n(z))$, and $w_j = f_j(z)$. The **holomorphic Jacobian matrix** of f is the matrix

$$\mathbf{J}_{\mathbb{C}} f = \frac{\partial(w_1, \ldots, w_n)}{\partial(z_1, \ldots, z_n)}.$$

Theorem 1.12. (Transformation formula of Bergman kernel) Suppose D_1, D_2 are domains in $\mathbb{C}^n$, and $f : D_1 \to D_2$ is a biholomorphism. Then

$$\det \mathbf{J}_{\mathbb{C}} f(z) K_{D_2}(f(z), f(w)) \det \overline{\mathbf{J}_{\mathbb{C}} f(w)} = K_{D_1}(z,w).$$

2 Minimal Domains

2.1 Definition and Properties

Definition 2.1. Let $D \subset \mathbb{C}^n$ be a domain, and $p \in D$. We say that D is a **minimal domain** with center p if

$$K_D(p,p) = \frac{1}{\mu(D)}.$$

Remark 2.2. To say that a domain $D \subset \mathbb{C}^n$ is a minimal domain with center p is equivalent to say that the equality in the diagonal estimate in Remark 1.10 is achieved at p.

There is an alternative definition of bounded minimal domains, which concerns the property of biholomorphisms on it:

Definition 2.3. (Alternative definition, [DT21]) Let $D \subset \mathbb{C}^n$ be a bounded domain. D is called a **minimal domain** with center $p \in D$ if $\mu(D) \leq V(D')$ for any biholomorphism $\varphi : D \to D'$ such that $\det \mathbf{J}_{\mathbb{C}}\varphi(p) = 1$.

Proposition 2.4. Suppose $D \subset \mathbb{C}^n$ is a bounded domain, and let $p \in D$. Write $H_p = \{f \in A^2(D) \mid f(p) = 0\}$, and let $T_p = (H_p)^\perp \subset A^2(D)$. Then:
(1) H_p is a Hilbert subspace of $A^2(D)$;
(2) $\dim_{\mathbb{C}}\ T_p = 1$;
(3) $K_D(z,p) \in T_p$.

Proof. To prove (1), it suffices to prove that H_p is closed, because $A^2(D)$ is a Hilbert space. Suppose $\{f_n\}$ is a sequence in H_p and $f_n \to f$ in $A^2(D)$. Applying Proposition 1.1 gives f_n converges to f locally uniformly over D. Since $f_n(p) = 0$ for all n, we have $f(p) = 0$, and therefore $f \in H_p$. This completes the proof of (1).

For (3), if $g \in H_p$, then by the reproducing property we have

$$\begin{aligned} 0 = g(p) &= \int_D g(w) K_D(p,w) d\mu(w) \\ &= \int_D g(w) \overline{K_D(w,p)} d\mu(w) \\ &= \langle g, K_D(\cdot, p) \rangle, \end{aligned}$$

which is what we want.

To prove (2), first notice that we have the following:

Claim. If $h_1, h_2 \in T_p$ and $h_1(p) = h_2(p)$, then $h_1 = h_2$ over D.

Proof of Claim. Let $\varphi = h_1 - h_2$. Then $\varphi(p) = 0$, so

$$\varphi \in H_p \cap T_p = H_p \cap (H_p)^\perp = \{0\}.$$

□

Suppose $h_1, h_2 \in T_p \setminus \{0\}$. Then $h_1(p), h_2(p)$ are nonzero, because $T_p \cap H_p = \{0\}$. Let $\lambda = \frac{h_1(p)}{h_2(p)}, \psi = \lambda h_2$. Observe that $\psi \in T_p$ as $h_2 \in T_p$, and

$$\psi(p) = \frac{h_1(p)}{h_2(p)} h_2(p) = h_1(p).$$

By the claim, this implies $\lambda h_2 = \psi = h_1$, so T_p has dimension at most 1. As we showed that $K_D(z,p) \in T_p$ and $K_D(z,p)$ is not the zero function, we conclude that $\dim_{\mathbb{C}} T_p = 1$. □

This implies the following equivalent characterization of bounded minimal domains.

Proposition 2.5. Let $D \subset \mathbb{C}^n$ be a bounded domain, and $p \in D$. The following are equivalent:

(1) D is a minimal domain with center p;

(2) $T_p = \{f \mid f \equiv C \text{ for some } C \in \mathbb{C}\}$;

(3) $K_D(z,p)$ is a constant function in z.

Proof. It is immediate from the previous proposition that (2) is equivalent to (3). To see that (1) implies (2), let

$$\{\mu(D)^{-\frac{1}{2}}\} \cup \{\varphi_j\}_{j=2}^{\infty}$$

be a complete orthonormal basis for $A^2(D)$. Then we have

$$K_D(z,z) = \frac{1}{\mu(D)} + \sum_{j=2}^{\infty} |\varphi_j(z)|^2$$

and

$$K_D(p,p) = \frac{1}{\mu(D)}.$$

Therefore $\varphi_m(p) = 0$ for all $m \geq 2$, so the subspace spanned by $\{\varphi_m\}$ is a subspace of H_p. On the other hand, every function $g \in H_p \subset A^2(D)$ is a countable linear combination of $\{\mu(D)^{-1/2}\} \cup \{\varphi_n\}_{n=2}^{\infty}$. As $g(p) = 0$, it follows g is a linear combination of φ_m's only. Thus H_p is spanned by $\{\varphi_m\}$, and the span of $\{\mu(D)^{-1/2}\}$ is the space of all constant functions (denote it by M). Since the basis is an orthogonal set, we conclude that $M \perp H_p$. Hence, $M \subset T_p$. As T_p is 1-dimensional, we have $M = T_p$.

To see that (3) implies (1), suppose $K_D(z,p)$ is constant. By conjugate symmetry of the Bergman kernel this implies $K_D(p,z)$ is also constant. Say $K_D(p,z) \equiv C$. Applying the reproducing property to the constant function $f \equiv \mu(D)^{-1}$ gives

$$\frac{1}{\mu(D)} = \int_D \frac{1}{\mu(D)} K_D(p,w)d\mu(w) = \int_D \frac{C}{\mu(D)} d\mu(w) = C.$$

Therefore, $K_D(p,p) = K_D(p,z) = C = \mu(D)^{-1}$, so D is a minimal domain with center p by definition. □

Example 2.6. Recall from Example 1.5 that the Bergman kernel of the unit ball $\mathbb{B}^n$ is given by

$$K_{\mathbb{B}^n}(z,w) = \frac{n!}{\pi^n} \frac{1}{(1 - z \cdot \overline{w})^{n+1}}.$$

Thus, $K(z,0)$ is a constant, so $\mathbb{B}^n$ is a minimal domain with center 0.

The center of a bounded minimal domain is unique.

Proposition 2.7. If D is bounded minimal domain with centers $p \in D$ and $q \in D$, then $p = q$.

Proof. If D is minimal with centers p and q, then $T_p = T_q = \{f \mid f \equiv C\}$. As H_p, H_q are closed subspace of $A^2(D)$, we have $(T_p)^\perp = ((H_p)^\perp)^\perp = H_p$, and similarly for H_q. Therefore, $H_p = (T_p)^\perp = (T_q)^\perp = H_q$. That is, if $f \in A^2(D)$, then $f(p) = 0$ if and only if $f(q) = 0$. But this could happen only when $p = q$. □

Proposition 2.8. Suppose D_1, D_2 are domains in $\mathbb{C}^n$, and $f : D_1 \to D_2$ is a biholomorphism such that $\det \mathbf{J}_{\mathbb{C}} f$ is constant. If D_1 is a minimal domain with center p, then D_2 is a minimal domain with center $f(p)$.

Proof. Since D_1 is a minimal domain with center p, $K_{D_1}(z, p)$ is constant. Applying the transformation formula of Bergman kernel gives

$$\det \mathbf{J}_{\mathbb{C}} f(z) K_{D_2}(f(z), f(p)) \det \overline{\mathbf{J}_{\mathbb{C}} f(p)} = K_{D_1}(z, p).$$

By assumption, f is a biholomorphism whose complex Jacobian has constant determinant. In particular, the determinant of its Jacobian never vanishes, so $K_{D_2}(f(z), f(p))$ is a constant. This proves that D_2 is a minimal domain with center $f(p)$. □

Definition 2.9. $f : \mathbb{C}^n \to \mathbb{C}^n$ is called a **shear** if there exists a holomorphic function $g : \mathbb{C}^{n-1} \to \mathbb{C}$ such that $g(0) = 0$, and $f(z, w) = (z, w + g(z))$ for all $z \in \mathbb{C}^{n-1}, w \in \mathbb{C}$.

Remark 2.10. In particular, shears are biholomorphisms with determinant of Jacobian being constant. Therefore, the image of a minimal domain under a shear is still a minimal domain.

Theorem 2.11. Let $D_1, D_2 \subset \mathbb{C}^n$ be bounded minimal domains with centers p_1, p_2 respectively. Suppose $f : D_1 \to D_2$ is a biholomorphism with $f(p_1) = p_2$. Then $\det \mathbf{J}_{\mathbb{C}} f$ is constant.

Proof. Since f is biholomorphic, for all $z, w \in D_1$ we have

$$\det \mathbf{J}_{\mathbb{C}} f(z) K_{D_2}(f(z), f(w)) \det \overline{\mathbf{J}_{\mathbb{C}} f(w)} = K_{D_1}(z, w). \tag{$*$}$$

Take $w = p_1$ in $(*)$ gives

$$\det \mathbf{J}_{\mathbb{C}} f(z) K_{D_2}(f(z), p_2) \det \overline{\mathbf{J}_{\mathbb{C}} f(p_1)} = K_{D_1}(z, p_1).$$

As D_1, D_2 are minimal domains with centers p_1, p_2 respectively, we have $K_{D_2}(f(z), p_2)$ and $K_{D_1}(z, p_1)$ are constant (and nonzero). Therefore, $\det \mathbf{J}_{\mathbb{C}} f(z)$ is also constant. □

Remark 2.12. Let $D_1, D_2 \subset \mathbb{C}$ be minimal domains both with center 0. Suppose $f : D_1 \to D_2$ is a biholomorphism with $f(0) = 0$. Then Theorem 2.11 implies that f is linear. However, this does not hold in $\mathbb{C}^n$ with $n > 1$ due to the existence of shear maps. Let $D_1 \subset \mathbb{C}^2$ be a minimal domain with center 0, and take $f(z, w) = (z, w + g(z))$ with a nonlinear holomorphic function g satisfying $g(0) = 0$

(for example, $g(z) = e^z - 1$). Then $f(D_1)$ is a minimal domain with center 0 by Proposition 2.8, and f is a biholomorphism from D_1 to its image.

For $D \subset \mathbb{C}^n$ a bounded domain, write $\mathrm{Aut}(D)$ for the group of all biholomorphisms $f : D \to D$. For $p \in D$, we further define the following subgroups of $\mathrm{Aut}(D)$:

$$A_p(D) = \{\varphi \in \mathrm{Aut}(D) \mid \varphi(p) = p\}$$
$$SA_p(D) = \{\varphi \in A_p(D) \mid \mathbf{J}_{\mathbb{C}}\varphi \equiv C,\ C \in \mathbb{C}\}.$$

Set

$$\mathcal{F}_p = \{f \in A^2(D) \mid f \circ \varphi = f,\ \forall\ \varphi \in SA_p(D)\}.$$

Notice that all constant functions are contained in $\mathcal{F}_p$.

Theorem 2.13. If $\mathcal{F}_p$ contains only constant functions, then D is a minimal domain with center p.

Proof. Fix $\varphi \in SA_p(D)$. By the transformation formula of Bergman Kernel, we have

$$K_D(z, w) = K_D(\varphi(z), \varphi(w))|\lambda|^2, \qquad (\star)$$

where $\lambda = \mathbf{J}_{\mathbb{C}}\varphi$ is a constant by definition of SA_p. Take $z = w = p$ in $(\star)$ gives $|\lambda| = 1$ because $\varphi \in SA_p(D) \subset A_p(D)$. Therefore, $K_D(z, w) = K_D(\varphi(z), \varphi(w))$. Let $K_p(z) = K_D(z, p)$, then $K_p \in A^2(D)$, and

$$(K_p \circ \varphi)(z) = K_D(\varphi(z), p) = K_D(\varphi(z), \varphi(p)) = K_D(z, p) = K_p(z).$$

Hence, $K_p \in \mathcal{F}_p$ by definition. By assumption $K_p(z) = K_D(z, p)$ must be a constant, which proves that D is a minimal domain with center p. □

Corollary 2.14. Suppose $G \subset \mathrm{GL}_n(\mathbb{C})$ is a subgroup and $D \subset \mathbb{C}^n$ is a domain invariant under the action of G and containing the origin. Define $\mathcal{F}(G, D) := \{f \in A^2(D) \mid f \circ T = f,\ \forall\ T \in G\}$. If $\mathcal{F}(G, D)$ contains only constant functions, then D is a minimal domain with center 0.

Proof. By assumption $G \subset SA_0(D)$, so $\mathcal{F}_0 \subset \mathcal{F}(G, D)$. Notice that $\mathcal{F}_0$ contains all constant functions, and the assumption gives $\mathcal{F}(G, D) = \{f \in A^2(D) \mid f \text{ is constant}\}$. Therefore, $\mathcal{F}_0$ contains only constant functions and the preceding theorem now gives the desired result. □

Remark 2.15. Notice that the converse of above corollary does not hold. Let $\mathbb{D} \subset \mathbb{C}$ be the unit disk, and consider $G = \{\mathrm{id}, -\mathrm{id}\}$. Then $\mathbb{D}$ is a minimal domain with center 0 and is invariant under G, but the function $f(z) = z^2$ is in $\mathcal{F}(G, \mathbb{D})$.

There is an open question regarding the exsitence of center-preserving biholomorphisms from a minimal domain to another minimal domain.

Question 2.16. Let D_1, D_2 be minimal domains with center p_1, p_2, respectively. Suppose that there exists a biholomorphism $g : D_1 \to D_2$, not necessarily $g(p_1) = p_2$. Does there always exist a biholomorphism $f : D_1 \to D_2$ such that $f(p_1) = p_2$?

2.2 Minimal Domains in $\mathbb{C}$

We introduce the classification of minimal domains in $\mathbb{C}$. This is done in [DT21].

Definition 2.17. $P \subset \mathbb{C}$ is said to be **polar** if there is a subharmonic function u, not identically $-\infty$, such that $P = \{z \in \mathbb{C} \mid u(z) = -\infty\}$.

Let $\Delta(z, r)$ denote the disk in $\mathbb{C}$ with center z and radius r.

Theorem 2.18. ([DT21], Theorem 1) Let $\Omega \subset \mathbb{C}$ be a domain, and let $K(z, w)$ be the Bergman kernel on Ω. Suppose there is a point $z_0 \in \Omega$ such that

$$K(z_0, z_0) = \frac{1}{\mu(\Omega)},$$

where if $\mu(\Omega) = \infty$ then $\mu(\Omega)^{-1} = 0$. Then:
(1) If $\mu(\Omega) = \infty$, then $\Omega = \mathbb{C} \setminus P$, where P is a closed polar set (possibly empty).
(2) If $\mu(\Omega) < \infty$, then $\Omega = \Delta(z_0, r) \setminus P$, where P is a polar set (possibly empty) closed in the relative topology of $\Delta(z_0, r)$, with $r = \sqrt{\mu(\Omega)\pi^{-1}}$.

Remark 2.19. [DT21] If P is a closed polar subset of a domain Ω, then $A^2(\Omega \setminus P) = A^2(\Omega)$.

Remark 2.20. Compact subsets of a polar set are totally disconnected. Therefore, if Ω is assumed to have smooth boundary in Theorem 2.18, then P is empty in the conclusion.

We end this section by remarking that one should not expect a similar classification to hold in $\mathbb{C}^n$ if $n > 1$. Indeed, the polydisc $\mathbb{D}^n$ is a Reinhardt domain (and is thus minimal, we will see this result in Section 4) which is not biholomorphic to the ball $\mathbb{B}^n$. However, $\mathbb{D}^n$ does not have smooth boundary, so one may ask instead if we could expect a result like Theorem 2.18 to hold if we assume the domain has smooth boundary. The answer turns out to be negative also in this case.

Example 2.21. ([DT21], Theorem 3) Let $D = \{z \in \mathbb{C}^2 \mid |z_1|^4 + |z_1|^2 + |z_2|^2 < 1\}$. Then D is complete Reinhardt, strongly convex, and not biholomorphic to $\mathbb{B}^2$.

Definition 2.22. [Sak89] Let $D \subset \mathbb{C}^n$ be a bounded domain with C^2 boundary. D is called **weakly pseudoconvex** if, for every p on the boundary of D and for every C^2 function ρ on an open neighborhood U of p in $\mathbb{C}^n$ such that $\mathrm{d}\rho(p) \neq 0$ and $D \cap U = \{z \in U \mid \rho(z) < 0\}$, we have

$$L[\rho; w](p) = \sum_{j,k} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p) w_j \overline{w_k} \geq 0$$

whenever w is non-zero with

$$\sum_j \frac{\partial \rho}{\partial z_j}(p) w_j = 0.$$

We say D is **strongly pseudoconvex** if we have $L[\rho; w](p) > 0$ with p, ρ, w same as above.

Theorem 2.23. [Bel81] A smooth bounded weakly pseudoconvex domain (that is not strongly pseudoconvex) cannot be biholomorphic to a strongly pseudoconvex domain.

Example 2.24. Let $\Omega = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^4 < 1\}$. Then Ω is weakly pseudoconvex, but not strongly pseudoconvex. In particular, Ω is not biholomorphic to the ball $\mathbb{B}^2$.

In general, one could consider a family of domains, called **Cartan egg-domains** or **Thullen domains**, given by

$$\Omega_m = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^{2m} < 1\}$$

for positive integers m. Ω_m are complete Reinhardt domains, so they are minimal domains. If $m_1 \neq m_2$, then Ω_{m_1} is not biholomorphic to Ω_{m_2}. This suggest that in $\mathbb{C}^n (n > 1)$, there are many families of minimal domains that are not biholomorphic to each other. Therefore, when $n > 1$, we want to study a larger class of domains that includes the collection of minimal domains in $\mathbb{C}^n$. This will be the main topic of Section 4.

3 Representations of Compact Lie Groups

In this section, we introduce some results about representations of compact Lie groups that will be useful for our discussion in Section 4.

3.1 Torus

Definition 3.1. A matrix Lie group T is a **torus** if it is isomorphic (as a Lie group) to a finite direct product of $S^1 \cong \mathrm{U}(1)$.

Theorem 3.2. ([Hal15], Theorem 11.2) Every connected, compact, abelian matrix Lie group is a torus.

Remark 3.3. Notice that the direct product of S^1 is connected, compact, and abelian, so the above theorem actually gives an alternative definition of the torus.

Definition 3.4. Let K be a compact Lie group. A subgroup T of K is a **torus** if T is isomorphic to a finite direct product of S^1. T is called a **maximal torus** if it is a torus and it is not contained in any other torus of K.

Example 3.5. A maximal torus of $\mathrm{U}(n)$ is given by

$$T = \left\{ \begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{bmatrix} \middle| |t_j| = 1, 1 \le j \le n \right\}.$$

Theorem 3.6. ([Hal15], Theorem 11.9) Let K be a compact Lie group. Then:

(1) If T_1 and T_2 are maximal tori of K, then there exists $x \in K$ such that $T_1 = xT_2x^{-1}$.

(2) Every element of K is contained in some maximal torus.

Theorem 3.7. ([HT94], Theorem A) Let G be a Lie group. Denote $\mathscr{C}_G$ the set of all compact subgroups of G, partially ordered by inclusion. Then every element in $\mathscr{C}_G$ is contained in a maximal element.

3.2 Representations of $\mathbb{T}^r$

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z} \cong S^1$ be the one dimensional torus. In this section we develop some properties of *complex* representations of $\mathbb{T}^r$.

Theorem 3.8. ([Bum13], Theorem 2.2) Suppose ρ and ρ' are irreducible representations of a group G on finite dimensional vector spaces V and V', respectively. If $L : V \to V'$ is a linear map, such that

$$\rho'(g)L = L\rho(g)$$

for all $g \in G$, then either L is an isomorphism, or $L = 0$.

Corollary 3.9. Suppose ρ is an irreducible representation of a group G on a finite dimensional vector space V such that $\rho(g)L = L\rho(g)$ for all $g \in G$. Then L is multiplication by a scalar.

Proof. Let λ be an eigenvalue of L. As the kernel of $L - \lambda I$ is nonempty (it contains the eigenvectors), by applying Schur's lemma to $L - \lambda I$ we see that $L - \lambda I = 0$. □

Corollary 3.10. Let G be a compact abelian group. Then any irreducible finite-dimensional representation of G is 1-dimensional.

Proof. Fix $h \in G$ and let $L = \rho(h)$. As G is abelian, we have $L\rho(g) = \rho(g)L$ for all $g \in G$. Applying Corollary 3.9 gives that $L = \rho(h)$ is a multiplication by scalar, so every one dimensional subspace is ρ-invariant. As ρ is an irreducible representation, it follows that ρ is 1-dimensional. □

In particular, because $\mathbb{T}^r$ is compact and abelian, all irreducible representations of $\mathbb{T}^r$ are one dimensional. Furthermore, we can explicitly compute the character of irreducible representations of $\mathbb{T}^r$.

Proposition 3.11. ([Bum13], Proposition 15.4) Every irreducible complex representation of $(\mathbb{R}/\mathbb{Z})^r$ is of form

$$(x_1, \ldots .x_r) \mapsto \exp(2\pi i \sum_{j=1}^{r} k_j x_j), x = (x_1, \ldots, x_r) \in [0,1)^r \cong (\mathbb{R}/\mathbb{Z})^r$$

for some $(k_1, \dots k_r) \in \mathbb{Z}^r$.

Remark 3.12. Therefore, the irreducible characters of $\mathbb{T}^r$ corresponds to the group $\mathbb{Z}^r$.

We have the following Maschke's theorem for compact Lie groups.

Theorem 3.13. ([Bum13], Proposition 2.2) If G is a compact Lie group, then every finite dimensional representation of G is a direct sum of irreducible representations.

Remark 3.14. Let ρ be a representation of $\mathbb{T}^r$ on a finite dimensional complex vector space V (say $\dim V = n$). As $\mathbb{T}^r$ is a compact Lie group, by the above results, we have

$$V = \bigoplus_{l=1}^{n} V_{k^{(l)}},$$

where $k^{(l)} = (k_1^{(l)}, \dots, k_r^{(l)}) \in \mathbb{Z}^r$, and $V_{k^{(l)}}$ is an irreducible representation of $\mathbb{T}^r$ as in Proposition 3.11, for all $1 \leq l \leq n$. Let A be the $n \times r$ matrix whose l-th row is $k^{(l)}$. By applying a change of basis if needed, we may assume that the matrix of $\rho(x)$ is given by

$$\rho(x) = \begin{bmatrix} \exp\left(2\pi i \sum_{j=1}^{r} k_j^{(1)} x_j\right) & & \\ & \ddots & \\ & & \exp\left(2\pi i \sum_{j=1}^{r} k_j^{(n)} x_j\right) \end{bmatrix} \tag{1}$$

for all $x \in (\mathbb{R}/\mathbb{Z})^r$.

Proposition 3.15. Let ρ, V, $V_{k^{(l)}}$, and A be the same as in Remark 3.14. If ρ is further assumed to be a faithful representation, then $\operatorname{rank} A = r$.

Proof. Since A is an $n \times r$ matrix, we only need to prove that $\ker A = \{0\}$. Suppose for contradiction that $x = (x_1, \dots, x_r) \in \mathbb{R}^r \setminus \{0\}$ satisfies $Ax = 0$. Then we may assume that $x \notin \mathbb{Z}^r$ by scaling. For $x \in \mathbb{R}^r$, define $\rho(x)$ using equation (1). Let $[x] \in [0,1)^r$ be the decimal part (taken component-wise) of x. Then $[x]$ is congruent to x modulo $\mathbb{Z}$, so $[x] \neq 0$ and $\rho(x) = \rho[x]$ because the exponential function in (1) is $\mathbb{Z}$-periodic. We compute

$$\begin{aligned} 0 = Ax &= \begin{bmatrix} k_1^{(1)} & \cdots & k_r^{(1)} \\ \vdots & \ddots & \vdots \\ k_1^{(n)} & \cdots & k_r^{(n)} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix} \\ &= \begin{bmatrix} \sum_{j=1}^{r} k_j^{(1)} x_j \\ \vdots \\ \sum_{j=1}^{r} k_j^{(n)} x_j \end{bmatrix}. \end{aligned}$$

Therefore, $\rho[x] = \rho(x) = I$. As ρ is faithful, we must have $[x] = 0$, so $x \in \mathbb{Z}^r$, which is a contradiction. Therefore we must have $\ker A = 0$ and hence $\operatorname{rank} A = r$.

□

4 Quasi-Reinhardt Domains

4.1 Original Definition

Definition 4.1. Let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be a representation of the group G. A holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$ is **ρ-invariant** if $f(\rho(g)z) = f(z)$ for all $g \in G$ and all $z \in \mathbb{C}^n$. Denote the algebra of ρ-invariant functions on $\mathbb{C}^n$ by $\mathcal{O}(\mathbb{C}^n)^\rho$. A domain $D \subset \mathbb{C}^n$ is **ρ-invariant** if $\rho(g)(D) = D$ for all $g \in G$.

We now give the definition of quasi-Reinhardt domains. Let $\mathbb{T}^r$ be the torus group of dimension $r \geq 1$. For $\mathbf{a}_i = (a_{i1}, \ldots, a_{ir}) \in \mathbb{Z}^r$ for $1 \leq i \leq n$, $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{T}^r$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, define $\lambda^{\mathbf{a}_i} = \prod_{j=1}^r \lambda_j^{a_{ij}}$, and define $\rho_A : \mathbb{T}^r \to \mathrm{GL}_n(\mathbb{C})$ to be the linear action given by

$$\rho_A(\lambda)z = (\lambda^{\mathbf{a}_1} z_1, \ldots, \lambda^{\mathbf{a}_n} z_n),$$

where A is the matrix (a_{ij}).

Definition 4.2. (Original definition in [LR19]) Suppose $D \subset \mathbb{C}^n$ is a domain. If there exists some ρ_A such that $\mathcal{O}(\mathbb{C}^n)^\rho = \mathbb{C}$ and D is ρ_A-invariant, then D is a **quasi-Reinhardt domain** of rank r with respect to ρ_A.

Remark 4.3. Throughout this section, we consider domains $D \subset \mathbb{C}^n$ that contains the origin.

Discussion 4.4. Definition 4.2 is the original definition of quasi-Reinhardt domain given in [LR19], and it is also discussed in [DR16]. In Section 4 of [DR16], the following claims are made: If $\rho : \mathbb{T}^r \to \mathrm{GL}_n(\mathbb{C})$ is a holomorphic linear action such that $\mathcal{O}(\mathbb{C}^n)^\rho = \mathbb{C}$ and D is a quasi-Reinhardt domain with respect to ρ, then (by considering the induced action of ρ on $A^2(D)$) there exists *unique* $\mathbf{a}_i \in \mathbb{Z}^r$ for $1 \leq i \leq n$ and a corresponding coordinate system $z = (z_1, \ldots, z_n)$, such that the representation can be written as

$$\rho(\lambda)(z) = (\lambda^{\mathbf{a}_1} z_1, \ldots, \lambda^{\mathbf{a}_n} z_n)$$

for every $\lambda \in \mathbb{T}^r$, and the $n \times r$ matrix A formed by $\mathbf{a}_i$ satisfies $\operatorname{rank} A = r$. However, we will see later that this claim is incorrect, and as a possible correction, we could add the condition $\operatorname{rank} A = r$ to the definition of quasi-Reinhardt domains. We shall also see that the definition of the rank of a quasi-Reinhardt domain should be modified.

Question 4.5. Suppose $\rho_A : \mathbb{T}^r \to \mathrm{GL}_n(\mathbb{C})$ is a representation of $\mathbb{T}^r$ as in Definition 4.2, how could one check if all ρ_A-invariant holomorphic functions on $\mathbb{C}^n$ are constant?

Theorem 4.6. Suppose $D \subset \mathbb{C}^n$ is a domain and $A \in \mathrm{Mat}_{n\times r}(\mathbb{Z})$. Let $\rho_A : \mathbb{T}^r \to \mathrm{GL}_n(\mathbb{C})$ be the associated torus representation. The following are equivalent:
(1) There exists a non-constant ρ_A-invariant holomorphic function on D.
(2) There exists a non-constant ρ_A-invariant holomorphic function on $\mathbb{C}^n$.

Proof. To show that (2) $\Rightarrow$(1), suppose there is a ρ_A-invariant holomorphic function f on $\mathbb{C}^n$. Then $f|_D$

is a ρ_A-invariant holomorphic function on D. Conversely, say $f \in \mathcal{O}(D)$ is ρ_A-invariant. As $0 \in D$, by [Hör66] Theorem 2.4.5 we have

$$f(z) = \sum b_{k_1,\ldots,k_n} z_1^{k_1} \cdots z_n^{k_n},$$

for unique constants $b_{k_1,\ldots,k_n}$, in some neighborhood of 0. Let A be an $n \times r$ matrix in integers with rank r, and denote its rows by $\mathbf{a}_j, 1 \le j \le n$. Then by ρ_A-invariance:

$$\sum b_{k_1,\ldots,k_n} z_1^{k_1} \cdots z_n^{k_n} = f(z) = f(\rho_A(\lambda)z) = \sum b_{k_1,\ldots,k_n} \lambda^{\mathbf{a}_1 k_1 + \cdots + \mathbf{a}_n k_n} z_1^{k_1} \cdots z_n^{k_n}. \tag{2}$$

Since f is not constant, there is a nonzero multi-index $(m_1, \cdots, m_n)$ such that $b_{m_1,\ldots,m_n} \neq 0$. By comparing coefficients in the above equality of series we have $\lambda^{\mathbf{a}_1 m_1 + \cdots + \mathbf{a}_n m_n} = 1$, so $g(z) = b_{m_1,\ldots,m_n} z_1^{m_1} \cdots z_n^{m_n}$ is a ρ_A-invariant holomorphic function on $\mathbb{C}^n$. □

Remark 4.7. The above proof could be modified to give an answer to Question 4.5. By using equation (2) in the above proof, we see that if $\mathbf{a}_i \in \mathbb{Z}^r$ for $1 \le i \le n$, then an entire function f is ρ_A-invariant if and only if

$$b_{k_1,\ldots,k_n} = b_{k_1,\ldots,k_n} \lambda^{\mathbf{a}_1 k_1} \cdots \lambda^{\mathbf{a}_n k_n} = b_{k_1,\ldots,k_n} \prod_{i=1}^{n} \prod_{j=1}^{r} \lambda_j^{k_i a_{ij}}$$

for any $\lambda \in \mathbb{T}^r$ and any non-zero multi-index $(k_1, \cdots, k_n)$. Thus, for given $\mathbf{a}_i$'s, if the equation $\mathbf{k}^t A = 0$ yields no solution $\mathbf{k} = (k_1, \ldots k_n) \in \mathbb{Z}^n \setminus \{0\}$ with $k_j \ge 0$, then f must be a constant function. A more useful case is, in particular, if $A \in \mathrm{Mat}_{n\times r}(\mathbb{Z})$ satisfies $a_{ij} > 0$ for all i, j, then all ρ_A-invariant holomorphic functions on $\mathbb{C}^n$ are constant.

We now look back at Discussion 4.4. Consider the following example:

Example 4.8. Consider the domain $\Omega = \{(z_1, z_2, z_1 z_2 + z_3) \mid z_j \in \mathbb{D}, 1 \le j \le 3\}$. Take $\mathbf{a}_1 = (1, 0), \mathbf{a}_2 = (0, 1), \mathbf{a}_3 = (1, 1)$. For $\lambda = (\lambda_1, \lambda_2) \in \mathbb{T}^2$ and $w = (w_1, w_2, w_3) \in \mathbb{C}^3$, we compute

$$\rho_A(\lambda)w = (\lambda^{\mathbf{a}_1} w_1, \lambda^{\mathbf{a}_2} w_2, \lambda^{\mathbf{a}_3} w_3) = (\lambda_1 w_1, \lambda_2 w_2, \lambda_1 \lambda_2 w_3).$$

By Remark 4.7, we see that $\mathcal{O}(\mathbb{C}^n)^{\rho_A} = \mathbb{C}$. Furthermore, set $w_1 = z_1, w_2 = z_2, w_3 = z_1 z_2 + z_3$, we have

$$\rho_A(\lambda)w = (\lambda_1 z_1, \lambda_2 z_2, \lambda_1 \lambda_2 (z_1 z_2 + z_3)) = (\widetilde{z_1}, \widetilde{z_2}, \widetilde{z_1}\widetilde{z_2} + \widetilde{z_3}),$$

where $\widetilde{z_1} = \lambda_1 z_1, \widetilde{z_2} = \lambda_2 z_2, \widetilde{z_3} = \lambda_1 \lambda_2 z_3$. As $\lambda_j \in \mathbb{D}$, we have Ω is ρ_A-invariant, so Ω is a quasi-Reinhardt domain of rank 2 with respect to ρ_A by Definition 4.2. Let's now consider $\mathbf{b}_1 = (1, 1, 0), \mathbf{b}_2 = (0, 0, 1), \mathbf{b}_3 = (1, 1, 1)$, and let B be the matrix whose rows are $\mathbf{b_j}$. Then for $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{T}^3$ and $w \in \mathbb{Z}^3$ same as above, we compute

$$\rho_B(\lambda)w = (\lambda_1 \lambda_2 z_1, \lambda_3 z_2, \lambda_1 \lambda_2 \lambda_3 (z_1 z_2 + z_3)) = (z_1', z_2', z_1' z_2' + z_3'),$$

where $z_1' = \lambda_1 \lambda_2 z_1, z_2' = \lambda_3 z_2, z_3' = \lambda_1 \lambda_2 \lambda_3 z_3$. Again because $\lambda_j \in \mathbb{D}$ we have Ω is ρ_B-invariant.

Therefore, Ω is also a quasi-Reinhardt domain of rank 3 with respect to ρ_B. This contradicts the claim by [DR16] in Discussion 4.4, because $\operatorname{rank} B = 2$ while B is a 3×3 matrix.

Remark 4.9. We also see from Example 4.8 that the rank of a quasi-Reinhardt domain could vary by choosing appropriate representations of torus of varying dimensions. In fact, by [LR19], if D is a quasi-Reinhardt domain of rank r as in Definition 4.2, then it is also of rank s for any $1 \leq s < r$. To better study the behavior of rank under biholomorphisms, we will also define the rank of a quasi-Reinhardt domain to be a fixed value, as we shall see in the following definition. However, it is worth noticing that the preceding conclusion of [LR19] regarding the rank still holds if we formulate it in a different way: if D is a quasi-Reinhardt domain of rank r, then for any $1 \leq s < r$, there is some representation $\rho : T^s \to \mathrm{GL}_n(\mathbb{C})$ such that $\mathcal{O}(\mathbb{C}^n)^\rho = \mathbb{C}$ and D is ρ-invariant.

4.2 A Modified Definition

With the example and remark in Section 4.1, we give a modified definition of quasi-Reinhardt domains.

Definition 4.10. (Modified definition) Suppose $D \subset \mathbb{C}^n$ is a domain, and r is an integer between 1 and n. If there exists $A \in \mathrm{Mat}_{n\times r}(\mathbb{Z})$ such that $\operatorname{rank} A = r$, $\mathcal{O}(\mathbb{C}^n)^{\rho_A} = \mathbb{C}$, and D is ρ_A-invariant, then D is a **quasi-Reinhardt domain** with respect to ρ_A. The **rank** of a quasi-Reinhardt domain D is defined to be the maximum of r ($1 \leq r \leq n$) such that there exists $A \in \mathrm{Mat}_{n\times r}(\mathbb{Z})$ with $\operatorname{rank} A = r$ and D is quasi-Reinhardt with respect to ρ_A.

Definition 4.11. A domain $D \subset \mathbb{C}^n$ is m-**quasi-circular** (where $m = (m_1, \ldots, m_n)$, m_j being positive integers) if for all $\theta \in \mathbb{R}$, D is invariant under the map $\rho_m(\theta) : D \to \mathbb{C}^n$ defined by

$$\rho_m(\theta)(z_1, \ldots, z_n) = (e^{im_1\theta} z_1, \ldots, e^{im_n\theta} z_n).$$

If $m_j = 1$ for all $1 \leq j \leq n$, then D is called a **circular domain**.

If we regard ρ_m as a map $\mathbb{T} \to \mathrm{GL}_n(\mathbb{C}), \theta \mapsto \rho_m(\theta)$, then any m-quasi-circular domain is a quasi-Reinhardt domain with respect to ρ_m. In particular, any circular domain is a quasi-Reinhardt domain.

Example 4.12. For a concrete example, $\mathbb{G}^2 = \{(z_1 + z_2, z_1 z_2) \mid z_1, z_2 \in \mathbb{D}\}$ is a $(1, 2)$-quasi-circular domain in $\mathbb{C}^2$. More generally, the symmetrized polydisc $\mathbb{G}^n = \{(p_{n,1}(z), \ldots, p_{n,n}(z)) \mid z \in \mathbb{D}^n\}$, where $p_{n,k}(z) = \sum_{1\leq j_1 < \ldots < j_k \leq n} z_{j_1} \cdots z_{j_k}$, is a $(1, \ldots, n)$-quasi-circular domain in $\mathbb{C}^n$, and is thus a quasi-Reinhardt domain.

Definition 4.13. A domain $D \subset \mathbb{C}^n$ is called a **Reinhardt domain** if $z = (z_1, \ldots, z_n) \in D$ implies that $(e^{i\theta_1} z_1, \ldots, e^{i\theta_n} z_n) \in D$ for all $\theta_1, \ldots, \theta_n \in \mathbb{R}$.

Therefore, Reinhardt domains are quasi-Reinhardt domains with respect to $\rho_I : \mathbb{T}^n \to \mathrm{GL}_n(\mathbb{C})$, with I being the n-dimensional identity matrix. Notice that this implies the rank of a Reinhardt domain (as a quasi-Reinhardt domain) is n, because the rank of a quasi-Reinhardt domain is at most n by definition.

Remark 4.14. We know that Reinhardt domains in $\mathbb{C}^n$ are quasi-Reinhardt domains of rank n. With the original definition in [DR16], quasi-circular domains are quasi-Reinhardt domains of rank 1. However, we do not have any assertion regarding the rank of quasi-circular domains with the modified definition. This is illustrated by the following: Let Ω and ρ_A be as in Example 4.8. The example shows that Ω is a quasi-Reinhardt domain with rank at least 2. On the other hand, Ω is also $(1, 1, 2)$-quasi-circular. Therefore, Ω is an example of quasi-circular domain whose rank as a quasi-Reinhardt domain (with the modified definition) is higher than 1.

4.3 An Intrinsic Extension

To determine if a domain D is a quasi-Reinhardt domain requires to check if there is a $\rho_A : \mathbb{T}^r \to \mathrm{GL}_n(\mathbb{C})$ such that $\operatorname{rank} A = r$, $\mathcal{O}(\mathbb{C}^n)^{\rho_A} = \mathbb{C}$, and that D is ρ_A-invariant. This task is not easy to perform in general. Furthermore, we need to consider all such representations in order to compute the rank of a quasi-Reinhardt domain. Therefore, we attempt to give an intrinsic definition. Namely, we consider the group of unitary automorphisms of D.

Lemma 4.15. ([GKK11], Theorem 1.3.4) Let Ω be a bounded domain in $\mathbb{C}^n$. If $\{f_j\}$ is a sequence in $\mathrm{Aut}(\Omega)$ which converges uniformly on compact subsets of Ω and if, for some $p_0 \in \Omega$, the limit $\lim_{j\to\infty} f_j(p_0)$ is a point in Ω, then the limit holomorphic mapping $f_0 : \Omega \to \overline{\Omega}$ has image precisely equal to Ω and $f_0 \in \mathrm{Aut}(\Omega)$.

Proposition 4.16. Suppose $D \subset \mathbb{C}^n$ is a bounded domain containing the origin. Let $\mathrm{UA}(D) = \mathrm{U}(n) \cap \mathrm{Aut}(D)$ be the group of unitary automorphisms of D. Then $\mathrm{UA}(D)$ is a closed subgroup of $\mathrm{U}(n)$, and is thus a Lie subgroup ([Lee13], Theorem 20.12). In particular, since $\mathrm{U}(n)$ is compact, so is $\mathrm{UA}(D)$.

Proof. Let $\{T_j\}$ be a sequence in $\mathrm{UA}(D)$, such that $T_j \to T$ in U(n). Then this convergence is also uniform because $\mathrm{U}(n)$ is finite dimensional, so all norms on $\mathrm{U}(n)$ are equivalent to each other. Notice that $0 \in D$, and $T(0) = 0$, so by Lemma 4.15 we have $T \in \mathrm{Aut}(D)$. Therefore, $T \in \mathrm{UA}(D)$, which is what we want. □

Definition 4.17. (Intrinsic definition) Suppose $D \subset \mathbb{C}^n$ is a domain containing the origin. We say that D is a quasi-Reinhardt domain of rank r if the following conditions hold:
(1) There exists a maximal torus T of $\mathrm{UA}(D)$ such that if $f : \mathbb{C}^n \to \mathbb{C}$ is a T-invariant holomorphic function (that is, $f \circ \varphi = f$ for all $\varphi \in T$), then f is constant;
(2) $\dim T = r$.

We wish to prove that this definition really agrees with the modified Definition 4.10.

Lemma 4.18. Let T be a torus in UA(D) of dimension r. Then there is $A \in \mathrm{Mat}_{n\times r}(\mathbb{Z})$ such that $\operatorname{rank} A = r$, and the associated torus representation $\rho_A : \mathbb{T}^r \to \mathrm{GL}_n(\mathbb{C})$ satisfies $T = \mathrm{Image}(\rho_A)$.

Proof. By definition, we know that T is isomorphic (as a Lie group) to $\mathbb{T}^r$ for some $r \in \mathbb{Z}$, $r \geq 1$. On the other hand, T is a subgroup of $\mathrm{UA}(D)$, so it is also a subgroup of $\mathrm{GL}_n(\mathbb{C})$. Define $\rho : \mathbb{T}^r \to \mathrm{GL}_n(\mathbb{C})$

to be the composition of these isomorphism and inclusions, so that the following diagram commutes:

$$\mathbb{T}^r \xrightarrow{\cong} T \hookrightarrow \mathrm{UA}(D) \hookrightarrow \mathrm{GL}_n(\mathbb{C}), \qquad \rho: \mathbb{T}^r \to \mathrm{GL}_n(\mathbb{C}).$$

Then ρ is a faithful representation of $\mathbb{T}^r$. By Proposition 3.15, there exists $A \in \mathrm{Mat}_{n\times r}(\mathbb{Z})$ such that $\operatorname{rank} A = r$ and $\rho = \rho_A$ up to a change of basis in $\mathbb{C}^n$.

□

Proposition 4.19. Definition 4.10 is equivalent to Definition 4.17.

Proof. We first prove that Definition 4.10 implies Definition 4.17. Suppose D is quasi-Reinhardt with respect to $\rho_A : \mathbb{T}^r \to \mathrm{GL}_n(\mathbb{C})$, and $\operatorname{rank} D = r$.

Claim. $\operatorname{rank} \mathrm{d}\rho_A(\lambda) = r$ for all λ. In particular, $\rho_A : \mathbb{T}^r \to \mathrm{GL}_n(\mathbb{C})$ is a smooth immersion.

Proof of claim. Suppose $\mathbf{a}_j = (a_{j1}, \ldots, a_{jn})$ is the the j-th row of A. Choose local coordinates $\lambda = (e^{2\pi i x_1}, \ldots, e^{2\pi i x_r}) = (\lambda_1, \ldots, \lambda_r)$, and identify the image $\rho_A(\lambda)$ with points in $\mathbb{C}^{n^2}$ by

$$\rho_A(\lambda) = \begin{bmatrix} \lambda^{\mathbf{a}_1} & & \\ & \ddots & \\ & & \lambda^{\mathbf{a}_n} \end{bmatrix}$$
$$\sim (\lambda^{\mathbf{a}_1}, \ldots, \lambda^{\mathbf{a}_n}, 0, \ldots, 0) \in \mathbb{C}^{n^2}.$$

Then ρ_A is obviously a smooth map, and the differential of ρ_A is given by

$$\mathrm{d}\rho_A(\lambda) = \begin{bmatrix} a_{11}\frac{\lambda^{\mathbf{a}_1}}{\lambda_1} & \cdots & a_{1r}\frac{\lambda^{\mathbf{a}_1}}{\lambda_r} \\ \vdots & \ddots & \vdots \\ a_{n1}\frac{\lambda^{\mathbf{a}_n}}{\lambda_1} & \cdots & a_{nr}\frac{\lambda^{\mathbf{a}_n}}{\lambda_r} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \sim \begin{bmatrix} a_{11}\lambda_2\cdots\lambda_r & \cdots & a_{1k}\lambda_1\cdots\widehat{\lambda_k}\cdots\lambda_r & \cdots & a_{1r}\lambda_1\cdots\lambda_{r-1} \\ \vdots & & \vdots & & \\ a_{n1}\lambda_2\cdots\lambda_r & \cdots & a_{nk}\lambda_1\cdots\widehat{\lambda_k}\cdots\lambda_r & \cdots & a_{nr}\lambda_1\cdots\lambda_{r-1} \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \tag{3}$$

where $\sim$ denotes the row equivalence of matrices, and $\lambda_1 \cdots \widehat{\lambda_k} \cdots \lambda_r$ indicates that the k-th component of λ is omitted in the product. This row equivalence is achieved by dividing the j-th row of $\mathrm{d}\rho_A(\lambda)$ $\lambda_1^{a_{j1}-1} \cdots \lambda_r^{a_{jr}-1}$. Notice that in the matrix after this division, the non-zero block is achived by multiplying $\lambda_1 \cdots \widehat{\lambda_k} \cdots \lambda_r$ on the k-th column of A. Thus $\operatorname{rank} \mathrm{d}\rho_A(\lambda) = \operatorname{rank} A = r$ for all λ, and ρ_A is thus a smooth immersion. □

As ρ_A is an immersion with constant rank, $T := \operatorname{Image} \rho_A$ is thus an immersed submanifold (and hence a Lie subgroup) of $\mathrm{U}(n)$ with dimension r. T is connected, compact, and abelian because $\mathbb{T}^r$ is, so it is also a torus. By the respective property of ρ_A, T is contained in $\mathrm{Aut}(D)$, and $\mathcal{O}(\mathbb{C}^n)^T = \mathbb{C}$. We only need to prove that T is a maximal torus in $\mathrm{UA}(D)$. If not, then by definition T is properly contained in some other torus T' of $\mathrm{UA}(D)$. Then $s := \dim T' > \dim T = r$. By Lemma 4.18, there exists $B \in \mathrm{Mat}_{n\times s}(\mathbb{Z})$ with $\operatorname{rank} B = s$, so that D is ρ_B-invariant and $\mathcal{O}(\mathbb{C}^n)^{\rho_B} = \mathbb{C}$. This contradicts the definition of r that it is the maximal number with this property. Conversely, if $T \subset \mathrm{UA}(D)$ is a maximal torus satisfying properties as in Definition 4.17, then using Lemma 4.18 we find a ρ_A as desired, and the rank of D in the sense of Definition 4.10 is r because T is a maximal torus. □

We have the following lemma follows from page 708 of [Hei92].

Lemma 4.20. Suppose D is a quasi-Reinhardt domain with respect to ρ. Then $\mathcal{O}(D)^\rho = \mathbb{C}$.

4.4 Mappings on Quasi-Reinhardt Domains

As remarked in Section 2.2, one should not expect a simple classification theorem for minimal domains to hold in $\mathbb{C}^n$ when $n > 1$. Therefore, we attempt to find and study a broader class of domains to reveal more general properties of minimal domains. Quasi-Reinhardt domains are considered exactly due to this reason.

Theorem 4.21. ([LR19], Theorem 1.3) Let $D \subset \mathbb{C}^n$ be a bounded quasi-Reinhardt domain containing the origin. Then D is a minimal domain with center 0.

Proof. Suppose D is quasi-Reinhardt with respect to $\rho_A : \mathbb{T}^r \to \mathrm{GL}_n(\mathbb{C})$, where $A \in \mathrm{Mat}_{n\times r}(\mathbb{Z})$ is rank r. Let $\mathbf{a}_j, 1 \le j \le n$ be the rows of A. Then $\rho_A(\lambda) \in \mathrm{GL}_n(\mathbb{C})$ is a biholomorphism from D to D. Let $K(z, w)$ be the Bergman kernel on D. By the transformation formula of Bergman kernel (Theorem 1.12), for all $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathbb{T}^r$, we have

$$[\det \mathbf{J}_{\mathbb{C}}\rho_A(\lambda)(z)]\, K(\rho_A(\lambda)z, \rho_A(\lambda)w)\overline{\det \mathbf{J}_{\mathbb{C}}\rho_A(\lambda)(w)} = K(z, w).$$

Notice that $\det \mathbf{J}_{\mathbb{C}}\rho_A(\lambda)(z) = \lambda^{\mathbf{a}_1} \cdots \lambda^{\mathbf{a}_n}$ is constant, and $|\lambda_k| = 1$ for all $1 \le k \le r$, so

$$\det \mathbf{J}_{\mathbb{C}}\rho_A(\lambda)(z)\overline{\det \mathbf{J}_{\mathbb{C}}\rho_A(\lambda)(w)} = \lambda^{\mathbf{a}_1} \cdots \lambda^{\mathbf{a}_n}\overline{\lambda^{\mathbf{a}_1} \cdots \lambda^{\mathbf{a}_n}} = 1.$$

Take $w = 0$ now gives

$$K(z, 0) = K(\rho_A(\lambda)z, 0)$$

because $\rho_A(w)(0) = 0$. Recall that $K(\cdot, w) \in A^2(D)$ for all $w \in D$, so $K(\cdot, w) \in \mathcal{O}(D)^{\rho_A}$. By Lemma 4.20 we have $K(z, 0)$ is constant, and thus D is a minimal domain by Proposition 2.5. □

The converse of Theorem 4.21, however, does not hold in general. Recall that the image of a minimal domain under a shear is still a minimal domain. We see that the same property does not have to hold for quasi-Reinhardt domains.

Lemma 4.22. ([LR19], Corollary 5.4) Let D and D' be bounded quasi-Reinhardt domains in $\mathbb{C}^n$ containing the origin. Suppose $f : D \to D'$ is a proper holomorphic map such that $f^{-1}(0) = 0$. Then f is a polynomial.

Example 4.23. Let $\mathbb{B}^2$ be the unit ball in $\mathbb{C}^2$. $\mathbb{B}^2$ is a Reinhardt domain, so it is also quasi-Reinhardt. Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be given by

$$f(z, w) = (z, w + e^z - 1).$$

Notice that f is a shear, so it is a biholomorphism. Its inverse is given by

$$f^{-1}(\xi, \eta) = (\xi, \eta - e^\xi + 1).$$

In particular, it is also a proper map. Let $D = f(\mathbb{B}^2) = \{(\xi, \eta) \in \mathbb{C}^2 \mid |\xi|^2 + |\eta - e^\xi + 1|^2 < 1\}$. We show that D is not a quasi-Reinhardt domain. Suppose for contradiction that D was a quasi-Reinhardt domain. Then as $f^{-1}(0) = 0$ we have f is a polynomial by Lemma 4.22, but f is obviously not a polynomial by its definition. Notice, however, that $\mathbb{B}^2$ is a minimal domain, so D is also a minimal domain as it is the image under the shear map f.

In the remaining of this section, we study the biholomorphisms between quasi-Reinhardt domains.

Proposition 4.24. Suppose D and D' are bounded quasi-Reinhardt domains. If there exists a unitary biholomorphism $f : D \to D'$, then $\operatorname{rank} D = \operatorname{rank} D'$.

Proof. Let T be a maximal torus of dimension r in $\mathrm{UA}(D)$, such that all T-invariant holomorphic functions on $\mathbb{C}^n$ are constant. As $f : D \to D'$ is a biholomorphism, we have

$$\mathrm{Aut}(D') = \{f \circ \alpha \circ f^{-1} \mid \alpha \in \mathrm{Aut}(D)\}.$$

Since f is unitary, for $\alpha \in \mathrm{Aut}(D)$, we have $\beta = f \circ \alpha \circ f^{-1} \in \mathrm{UA}(D')$ if and only if $\alpha \in \mathrm{UA}(D)$, and $T' = f \circ T \circ f^{-1} := \{f \circ \varphi \circ f^{-1} \mid \varphi \in T\}$ is a maximal torus of dimension r in $\mathrm{UA}(D')$.Furthermore, if $g \in \mathcal{O}(D)^{T'}$, then

$$g \circ f \circ \varphi \circ f^{-1} = g$$

for all $\varphi \in \mathrm{UA}(D)$, which is equivalent to

$$g \circ f \circ \varphi = g \circ f$$

for all $\varphi \in \mathrm{UA}(D)$. Therefore, $g \circ f \in \mathcal{O}(D)^T$, and is therefore a constant by identifying T as the image of some ρ_A and applying Theorem 4.6. As f is biholomorphic, g must be a constant, so $\mathcal{O}(D)^{T'} = \mathbb{C}$. By identifying T' with the image of some $\rho_{A'}$ and applying Theorem 4.6 again, we conclude that $\mathcal{O}(\mathbb{C}^n)^{T'}$ is constant. □

In the setting of Proposition 4.24, if we only assume D and D' are merely biholomorphic (so, without assuming the existence of a unitary biholomorphism), then the rank is in general not preserved.

Let $\mathbb{B}^n$ be the unit ball in $\mathbb{C}^n$, and fix $a \in \mathbb{B}^n$. Define $P_0 = 0$

$$P_a(z) = \frac{\langle z, a\rangle}{\langle a, a\rangle} a \qquad \text{if } \ a \neq 0$$

and $Q_a = I - P_a$ where I is the identity map on $\mathbb{C}^n$. Let $s_a = (1 - |a|^2)^{1/2}$ and define

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle}.$$

Notice that similar to the 1-dimensional case, we have $\varphi_a \in \mathrm{Aut}(\mathbb{B}^n)$. The following lemma concerning the automorphisms of $\mathbb{B}^n$ will be used to construct an example of two biholomorphic quasi-Reinhardt domains with different rank.

Lemma 4.25. [Rud08] If $\psi \in \mathrm{Aut}(\mathbb{B}^n)$ and $a = \psi^{-1}(0)$, then there exists a unique $U \in \mathrm{U}(n)$ such that

$$\psi = U\varphi_a.$$

Remark 4.26. We compute

$$\varphi_0(z) = \frac{0 - P_0(z) - s_0 Q_0(z)}{1 - \langle z, 0\rangle} = -I.$$

In particular, if $\psi \in \mathrm{Aut}(\mathbb{B}^n)$ satisfies $\psi(0) = 0$, then $\psi \in \mathrm{U}(n)$.

Example 4.27. Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be the shear map defined by

$$f(z, w) = (z, w + z^2).$$

Its inverse is given by

$$f^{-1}(\xi, \eta) = (\xi, \eta - \xi^2).$$

Set $D = f(\mathbb{B}^2) = \{(\xi, \eta) \in \mathbb{C}^2 \mid |\xi|^2 + |\eta - \xi^2|^2 < 1\}$. $\mathbb{B}^2$ is a Reinhardt domain in $\mathbb{C}^2$, so it is a quasi-Reinhardt domain of rank 2. We show that D is a quasi-Reinhardt of rank 1 by computing $\mathrm{UA}(D)$ and using Definition 4.17. Let $\varphi \in \mathrm{UA(D)} \subset \mathrm{U}(2)$. Then there exists $\psi \in \mathrm{Aut}(\mathbb{B}^2)$ such that $\varphi = f \circ \psi \circ f^{-1}$. As $\varphi \in \mathrm{U}(2)$, we have $\varphi(0) = 0$. Also notice that $f(0) = 0$, so it follows that $\psi(0) = 0$. By Lemma 4.25, $\psi \in \mathrm{U}(2)$. Suppose

$$\psi = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

For $(\xi, \eta) \in D$, we compute

$$\begin{aligned} \varphi(\xi, \eta) &= f \circ \psi \circ f^{-1}(\xi, \eta) \\ &= f \circ \psi(\xi, \eta - \xi^2) \\ &= f(a\xi + b(\eta - \xi^2), c\xi + d(\eta - \xi^2)) \\ &= (a\xi + b(\eta - \xi^2), cz + d(\eta - \xi^2) + (a\xi + b(\eta - \xi^2))^2). \end{aligned}$$

As φ is linear, by looking at the first component of φ we must have $b = 0$. Then

$$\varphi(\xi, \eta) = (a\xi, c\xi + d(\eta - \xi^2) + a^2\xi^2).$$

As ψ is unitary, we must have that $|a| = |d| = 1$ (in particular, a, d are non zero), and that $c = 0$. Therefore, for φ to be linear, it must follow that $d = a^2$. This implies

$$\psi = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{2i\theta} \end{bmatrix}$$

for some $\theta \in \mathbb{R}$. Furthermore, by the above computation,

$$\varphi(\xi, \eta) = (e^{i\theta}\xi, e^{2i\theta}\eta) = \psi(\xi, \eta).$$

Therefore, we conclude that

$$\mathrm{UA(D)} = \left\{ \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{2i\theta} \end{bmatrix} \;\middle|\; \theta \in \mathbb{R} \right\}.$$

Therefore, UA(D) itself is a torus, and $\dim \mathrm{UA}(D) = 1$. This implies that $\operatorname{rank} D = 1$, and that f maps a quasi-Reinhardt domain of rank 2 to a quasi-Reinhardt domain of rank 1.

References

[Bel81] Steven R. Bell. Biholomorphic mappings and the $\bar{\partial}$-problem. *Ann. of Math. (2)*, 114(1):103–113, 1981.

[Bum13] Daniel Bump. *Lie groups*, volume 225 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.

[DR16] Fusheng Deng and Feng Rong. On biholomorphisms between bounded quasi-Reinhardt domains. *Ann. Mat. Pura Appl. (4)*, 195(3):835–843, 2016.

[DT21] Robert Xin Dong and John Treuer. Rigidity theorem by the minimal point of the Bergman kernel. *J. Geom. Anal.*, 31(5):4856–4864, 2021.

[GKK11] Robert E. Greene, Kang-Tae Kim, and Steven G. Krantz. *The geometry of complex domains*, volume 291 of *Progress in Mathematics*. Birkhäuser Boston, Ltd., Boston, MA, 2011.

[Hal15] Brian Hall. *Lie groups, Lie algebras, and representations*, volume 222 of *Graduate Texts in Mathematics*. Springer, Cham, second edition, 2015. An elementary introduction.

[Hei92] Peter Heinzner. On the automorphisms of special domains in $\mathbb{C}^n$. *Indiana Univ. Math. J.*, 41(3):707–712, 1992.

[Hör66] Lars Hörmander. *An introduction to complex analysis in several variables*. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1966.

[HT94] Karl H. Hofmann and Christian Terp. Compact subgroups of Lie groups and locally compact groups. *Proc. Amer. Math. Soc.*, 120(2):623–634, 1994.

[Kra13] Steven G. Krantz. *Geometric analysis of the Bergman kernel and metric*, volume 268 of *Graduate Texts in Mathematics*. Springer, New York, 2013.

[Lee13] John M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.

[LR19] Fengbai Li and Feng Rong. Remarks on quasi-Reinhardt domains. *Proc. Roy. Soc. Edinburgh Sect. A*, 149(2):297–304, 2019.

[Rud08] Walter Rudin. *Function theory in the unit ball of* $\mathbb{C}^n$. Classics in Mathematics. Springer-Verlag, Berlin, 2008. Reprint of the 1980 edition.

[Sak89] Akira Sakai. A characterization of weak pseudoconvexity. *Proc. Amer. Math. Soc.*, 105(2):314–316, 1989.