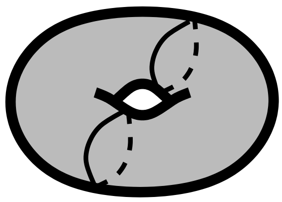
39. Fall 2022

Three-hour exam. Do as many questions as you can. Each is worth 4 marks. Please write clear maths and clear English which could be understood by one of your fellow students - pictures aid explanation but should not replace it! Include as much detail as is appropriate; you can use standard results and theorems in your answers provided you refer to them clearly.

1. Let $G = \langle a, b \rangle$ denote the free group on two letters, and let $H \leq G$ be the subgroup generated by the elements $\{x_i\}_{i \in \mathbb{Z}}$, where $x_i = a^i b^{-1} a^{1-i}$. Show that H is free.

2. Let X be the space obtained from a torus by attaching discs along the two curves shown below. Find the fundamental group of X, and identify the homology group $H_2(\tilde{X})$ of the universal cover of X as a module over the group ring $\mathbb{Z}[\pi_1(X)]$.



3. Let K be the Klein bottle. Compute the cohomology ring $H^*(K \times S^1; \mathbb{Z}_2)$.

4. Show that a connected closed non-orientable 3-manifold must have infinite fundamental group.

5. Consider the circle in the form of the abelian group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Show that there is a long exact sequence relating homology with coefficients in $\mathbb{Z}, \mathbb{R}$ and $\mathbb{T}$ and use it to compute $H_*(\mathbb{R}P^\infty; \mathbb{T})$. (You can assume the usual cellular chain complex for $\mathbb{R}P^\infty$.)

6. Let M^3 be a *homology sphere*: a connected closed compact 3-manifold with the same homology groups as S^3. Calculate the fundamental group and homology of the suspension ΣM. Use this to show that the suspension is homotopy-equivalent to S^4.

7. Let K be a (perhaps knotted) subspace of S^5 which is homeomorphic to the 3-sphere. Let N be a closed regular neighbourhood of K, so that N is a compact 5-manifold-with-boundary and is homotopy-equivalent to K. Let X be S^5 minus the interior of N, so that X is also a compact 5-manifold-with-boundary. By considering the relative cohomology $H^*(S^5, N)$ and applying excision and Lefschetz duality, calculate the homology of X.

8. Let X be the space obtained by gluing pairs of faces of a standard cube I^3 as shown. Compute the homology $H_*(X; \mathbb{Z})$.