Name: ____________________

PID: ____________________

Question	Points	Score
1	8	
2	7	
3	7	
4	6	
5	6	
6	6	
7	10	
Total:	50	

1. Write your name on the front page of your exam.
2. Read each question carefully, and answer each question completely.
3. Write your solutions clearly in the exam sheet.
4. Show all of your work; no credit will be given for unsupported answers.
5. You may use the result of one part of the problem in the proof of a later part, even you do not complete the earlier part.

1. (8 points) Prove that, up to isomorphism, there exist exactly two groups of order $3 \cdot 5 \cdot 13$.

2. Suppose G is a finite group. Let $\mathrm{Syl}_p(G)$ be the set of all Sylow p-subgroups of G. Suppose $P_1, P_2 \in \mathrm{Syl}_p(G)$ are distinct and $P_1 \cap P_2$ is minimal among all the subgroups that are the intersection of two distinct Sylow p-subgroups. Suppose N is a subgroup of $P_1 \cap P_2$ and $N \trianglelefteq P_i$ for $i = 1, 2$, and let $H := N_G(N)$.

(a) (2 points) Prove that for every $P \in \mathrm{Syl}_p(G)$, there exists $h \in H$ such that $hPh^{-1} \cap H \subseteq P_1$.

(b) (1 point) Prove that for every $P \in \mathrm{Syl}_p(G)$, there exists $h \in H$ such that $hPh^{-1} \cap P_2 = P_1 \cap P_2$.

(c) (2 points) Prove that for every $P \in \mathrm{Syl}_p(G)$, $N \subseteq P$.

(d) (2 points) Suppose P_1 is abelian. Prove that $P_1 \cap P_2 = \bigcap_{P \in \mathrm{Syl}_p(G)} P$.

3. A subgroup L of $\mathbb{Q}^n$ is called a *lattice* if it is finitely generated and its $\mathbb{Q}$-span is $\mathbb{Q}^n$.

   (a) (3 points) Prove that for every lattice L of $\mathbb{Q}^n$, there exists $x \in \mathrm{GL}_n(\mathbb{Q})$ such that $L = x\mathbb{Z}^n$.

   (b) (1 point) Suppose $L_1, \ldots, L_m$ are lattices in $\mathbb{Q}^n$. Prove that $L_1 + \cdots + L_m$ is also a lattice in $\mathbb{Q}^n$.

   (c) (3 points) Prove that for every finite subgroup G of $\mathrm{GL}_n(\mathbb{Q})$, there exists $x \in \mathrm{GL}_n(\mathbb{Q})$ such that $xGx^{-1} \subseteq \mathrm{GL}_n(\mathbb{Z})$.
   (**Hint.** Think about a lattice which is G-invariant.)

4. (6 points) Prove that if M is a projective A-module, then it is flat.

5. Let A be a unital commutative ring, $\operatorname{Spec}(A)$ denote the set of its prime ideals, and let M be an A-module.

   (a) (3 points) Suppose N_1 and N_2 are two submodules of M such that

   $$N_1 \simeq A/P_1 \quad \text{and} \quad N_2 \simeq A/P_2$$

   as A-modules for some P_1, $P_2 \in \operatorname{Spec}(A)$. Prove that if $P_1 \neq P_2$, then $N_1 \cap N_2 = \{0\}$.
   (**Hint.** Consider $\operatorname{ann}(x)$ for $x \in N_i$.)

   (b) (3 points) Suppose A is Noetherian. Prove that there exist a submodule N of M and $P \in \operatorname{Spec}(A)$ such that $N \simeq A/P$.
   (**Hint.** Consider $\Sigma := \{\operatorname{ann}(x) \mid x \in M \setminus \{0\}\}$.)

6. (6 points) Let E/F be a finite Galois extension with Galois group $G = \mathrm{Gal}(E/F)$, and let $[G, G]$ denote the derived subgroup of G. Define the subfield

$$F^{\mathrm{ab}} := \{e \in E \mid \theta(e) = e \text{ for all } \theta \in [G, G]\}.$$

That is, F^{ab} is the fixed field of $[G, G]$.

Prove that an element $e \in E$ lies in F^{ab} if and only if the field extension $F[e]/F$ is Galois and its Galois group $\mathrm{Gal}(F[e]/F)$ is abelian.

7. Let n be a positive integer, p a prime, $p \nmid n$, and $\Phi_n(x)$ the n-th cyclotomic polynomial. Let $E_{n,p}$ denote the splitting field of $\Phi_n(x)$ over $\mathbb{F}_p$.

   (a) (2 points) Prove that if $\zeta \in E_{n,p}$ is a zero of $\Phi_n(x)$, then the multiplicative order of ζ is n.

   (b) (3 points) Prove that $\mathrm{Gal}(E_{n,p}/\mathbb{F}_p)$ can be embedded into the group of automorphisms of $\langle\zeta\rangle$, where ζ is the element given in Part (a).

(c) (2 points) Prove that $\mathrm{Gal}(E_{n,p}/\mathbb{F}_p) \simeq \langle p + n\mathbb{Z} \rangle \subseteq (\mathbb{Z}/n\mathbb{Z})^\times$.

(d) (3 points) Prove that if n has two distinct odd prime factors, then $\Phi_n(x)$ is reducible over $\mathbb{F}_\ell$ for every prime ℓ.

Good Luck!

(Empty page)

(Empty page)

(Empty page)