Name: ______________________________ S.I.D.: ______________________________

Qualifier Exam in Applied Algebra

September 10, 2025

	Full	Real
# 1	10	
# 2	10	
# 3	10	
# 4	10	
# 5	10	
# 6	10	
# 7	10	
# 8	10	
# 9	10	
# 10	10	
Total	100	

Notes: 1) For computational questions, no credit will be given for unsupported answers 2) For proof questions, no credit will be given for no reasons or wrong reasons.

1. (10 points) Given $n \geq 3$, fix $A \in M_n(\mathbb{C}) = \mathbb{C}^{n\times n}$ satisfying that there exist two eigenvalues α, β of A with $\alpha \neq \beta$. Prove there exists unitary $Q \in M_n(\mathbb{C})$ such that

$$Q^H A Q = \begin{bmatrix} \alpha & \delta & v^H \\ 0 & \beta & 0 \\ 0 & w & B \end{bmatrix},$$

for some $\delta \in \mathbb{C}$, $v, w \in \mathbb{C}^{n-2}$, and $B \in M_{n-2}(\mathbb{C})$.

(Notationally: for all $m, n \geq 1$, $z^H = \bar{z}^T$ for all $z \in M_{m,n}(\mathbb{C}) = \mathbb{C}^{m\times n}$)

2. (10 points) Given $n \geq 1$, let $A \in M_n(\mathbb{C}) = \mathbb{C}^{n \times n}$. Prove if $x^H Ax = 0$, for all $x \in \mathbb{C}^n$, then $A = 0$. (Notationally: for all $m, n \geq 1$, $z^H = \bar{z}^T$ for all $z \in M_{m,n}(\mathbb{C}) = \mathbb{C}^{m \times n}$)

3. (10 points) Consider complex-valued square matrices A satisfying:

   - A has exactly two distinct eigenvalues of -2 and -3, with algebraic multiplicities of 6 and 1, respectively;
   - $\operatorname{rank}((A + 2I)^3) = 2$;

   Now considering all possible Jordan canonical forms similar to A, determine and write down one, and only one, of these from each similarity class.

4. (10 points) Given $n \geq 1$ and $\|\cdot\|$ a matrix norm of $M_n(\mathbb{C}) = \mathbb{C}^{n\times n}$ that is consistent (meaning $\|AB\| \leq \|A\|\|B\|$), let $A \in M_n(\mathbb{C})$.

   (a) Find a vector norm $\|\cdot\|'$ of $\mathbb{C}^n$ that is compatible with $\|\cdot\|$ (meaning $\|Ax\|' \leq \|A\|\|x\|'$).

   (b) Prove, for all eigenvalues λ of A, that $|\lambda| \leq \|A\|$.

5. (10 points) Let $\mathcal{A}$ be a unital associative algebra over $\mathbb{C}$ equipped with an antilinear and antimultplicative involution $A \mapsto A^*$. We say that $A \in \mathcal{A}$ is normal if A and A^* commute (example: normal matrices are normal elements of $\mathbb{C}^{N \times N}$). Prove that A is normal if and only if $A = X + iY$ with $X, Y \in \mathcal{A}$ commuting ($XY = YX$) and selfadjoint ($X^* = X$, $Y^* = Y$).

6. (10 points) Let (V, φ) be a finite-dimensional complex representation of a finite group G, and let $\chi(g) = \operatorname{Tr} \varphi(g)$ be its character. Prove that $|\chi(g)| \leq \dim \mathrm{V}$ for all $g \in \mathrm{G}$, and that the bound is sharp.

7. (10 points) Let (V, φ) be a finite-dimensional irreducible complex representation of a finite group G, and let χ be its character. Show that for any $g_1, g_2 \in \mathrm{G}$ we have $\frac{1}{|G|} \sum_{h \in \mathrm{G}} \chi(g_1 h g_2 h^{-1}) = \frac{\chi(g_1)\chi(g_2)}{\dim \mathrm{V}}$.

8. (10 points) Let (V, φ) be a complex representation of the symmetric group S_n, and let χ be its character. Prove that $\chi(g) \in \mathbb{R}$ for all $g \in \mathrm{S}_n$.

9. (10 points) Let k be an algebraically closed field and let $I \subseteq k[x, y]$ be an ideal. Suppose that the variety of I is $\mathbf{V}(I) = \{(1, 0), (0, 1)\} \subseteq k^2$.

   (a) Prove that $k[x, y]/I$ is a finite-dimensional k-vector space.

   (b) Do we necessarily have $\dim_k(k[x, y]/I) < 100$?

10. (10 points) Find two finite matrix groups $G, H \subseteq GL_2(\mathbb{C})$ such that ...

- G and H are isomorphic as abstract groups, but
- the Hilbert series of the graded rings $\mathbb{C}[x,y]^G$ and $\mathbb{C}[x,y]^H$ are different.