Numerical Analysis Qualifying Examination

Instructors: Ioana Dumitriu, Melvin Leok

Fall 2025

NAME: ______________________________

Part I: Math 270A

Problem	Points Possible	Points Earned
1	25	
2	25	
3	25	
Total	75	

1. (25 points) Let A be a nonsingular real $n \times n$ matrix that has an LU decomposition $A = LU$, where L is a unit lower triangular matrix and U is an upper triangular one.

a) Show that this decomposition is unique.

b) Suppose now that we are given **only** $A^T A$ (not A!), and we are aiming to find the upper triangular factor R with $R(i, i) > 0$ in the QR decomposition of A. Explain how we can do it by describing an algorithm for it (you don't have to write pseudocode).

c) Suppose now we are given **only** A. Consider now the alternate way of computing R from A, by first **efficiently** obtaining $A^T A$ and then applying the method you devised in b).

What is the complexity of this new algorithm? Compare it with the complexity of computing R through the QR decomposition of A (**most efficiently**).

2. (25 points) Let A be a $n \times n$ Hermitian matrix.

a) Describe an algorithm to reduce A to tridiagonal form via a sequence of unitary similarity transformations. (You don't have to write pseudocode.) Explain why the result is **tridiagonal, real, and symmetric**.

b) Let v be an $n \times 1$ vector. Prove that the eigenvalues of A interlace with those of $A + vv^*$, as follows:

(i) Assume first A is positive definite, so that $A^{1/2}$ exists and is well-defined. Write $A + vv^* = CC^*$, where C is an explicit $n \times (n+1)$ matrix involving $A^{1/2}$ and v.

(ii) Under the same conditions, use the Cauchy Interlacing Theorem for a Hermitian matrix and one of its principal $(n-1) \times (n-1)$ submatrices to argue that the eigenvalues of A and those of $A + vv^*$ interlace.

(iii) Now extend the result to A Hermitian (not necessarily positive definite).

3. (25 points)

a) Recall the convergence of the Conjugate Gradient (CG) method. Suppose that we applied it to a positive definite matrix A and vector b, and suppose that the error at step 0 were $||e_0||_A = 1$ and the error at step 20 was $||e_{20}||_A = 2 \cdot 4^{-10}$. Here, $||x||_A = \sqrt{x^T Ax}$.

Provide a bound for the condition number $\kappa(A)$ (corresponding to the 2-norm).

b) Although the CG method is only applicable to positive definite matrices, one can use it on general, full-rank matrices in conjunction with the normal equations (CGN): instead of solving $Ax = b$, solve $A^* Ax = A^* b$.

Explain why this may be undesirable.

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Part II: Math 270BC

Problem	Points Possible	Points Earned
4	25	
5	25	
6	25	
7	25	
8	25	
Total	125	

4. (25 points) Prove that if D is a compact set in $\mathbb{R}^d$, then the set of all the polynomials in D is dense in $C(D)$. You may use the Stone–Weierstrass theorem, which states that if $D \subset \mathbb{R}^d$ is a compact set and S is a subspace of $C(D)$ with the following properties:

(a) S contains all constant functions.

(b) $u, v \in S$ implies $uv \in S$.

(c) For each pair of points $x, y \in D$, $x \neq y$, there exists $v \in S$ such that $v(x) \neq v(y)$.

Then S is dense in $C(D)$, i.e., for any $v \in C(D)$, there is a sequence $\{v_n\} \subset S$ such that

$$\|v - v_n\|_{C(D)} \to 0, \qquad \text{as } n \to \infty.$$

5. (25 points) Recall that $B_i^k = V_i^k B_i^{k-1} + (1 - V_{i+1}^k) B_{i+1}^{k-1}$, where $V_i^k(x) = \frac{x - t_i}{t_{i+k} - t_i}$, and

$$B_i^0(x) = \begin{cases} 1 & \text{if } t_i \le x < t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

Let $U_i^k(s) = (t_{i+1} - s)(t_{i+2} - s) \cdots (t_{i+k} - s)$, where $U_i^0(s) = 1$. Prove that

$$U_i^k(s) V_i^k(x) + U_{i-1}^k(s)[1 - V_i^k(x)] = (x - s) U_i^{k-1}(s).$$

Hint: Fix s and observe that both sides are linear functions of x.

Then prove that,

$$\sum_{i=-\infty}^{\infty} U_i^k(s) B_i^k(x) = (x - s) \sum_{i=-\infty}^{\infty} U_i^{k-1}(s) B_i^{k-1}(x).$$

Finally, use this to show that,

$$\sum_{i=-\infty}^{\infty} U_i^k(s) B_i^k(x) = (x - s)^k.$$

6. (25 points) Suppose that we wish to approximate an even function by a polynomial of degree$\leq n$ using the norm $\|f\| = (\int_{-1}^{1} |f(x)|^2 dx)^{1/2}$. Prove that the best approximation is also even.

7. (25 points) Determine the order of the two-step method

$$y_{n+2} - y_n = \frac{2}{3}h[f(t_{n+2}, y_{n+2}) + f(t_{n+1}, y_{n+1}) + f(t_n, y_n)], \qquad n = 0, 1, \ldots$$

Is it A-stable?

8. (a) (15 points) Construct the three-stage collocation Runge–Kutta method for collocation points, 0, 1/2, and 1, and express it in the form of a Butcher tableau. (Show all work, including the integrals, no credit will be awarded for simply writing down the coefficients)

(b) (10 points) Prove that the resulting Lobatto IIIA Runge–Kutta method is fourth-order accurate. (You may use the theorem relating the order of collocation methods and the orthogonality properties of the polynomial with the collocation points as its roots)