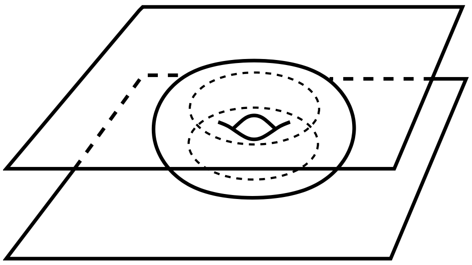
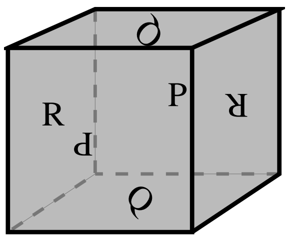
46. Fall 2025

Three-hour exam. Do as many questions as you can. Each is worth 4 marks. Please write clear maths and clear English which could be understood by one of your fellow students - pictures aid explanation but should not replace it! Include as much detail as is appropriate; you can use standard results and theorems in your answers provided you refer to them clearly. Notes may not be used.

1. Let X be the result, shown below, of "sandwiching" a standard 2-torus between two infinite horizontal planes in $\mathbb{R}^3$, so that each plane is tangent to the torus along a circle. Compute the integer homology groups $H_*(X;\mathbb{Z})$.



2. Let X be the space obtained by gluing opposite pairs of faces of a standard cube I^3 via 180 degree rotations, as shown. Compute the homology $H_*(X;\mathbb{Z})$.



3. Let X be a space whose homology is given by

$$H_k(X;\mathbb{Z}) = \begin{cases} \mathbb{Z}_4 & \text{if } k=2 \\ \mathbb{Z} & \text{if } k=0 \\ 0 & \text{otherwise.} \end{cases}$$

Compute $H_*(\mathbb{R}P^2 \times X;\mathbb{Z})$ and $H_*(\mathbb{R}P^2 \times X;\mathbb{Z}_2)$.

4. Compute $\mathrm{Ext}(\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3, \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_6)$. (Here, Ext denotes $\mathrm{Ext}^1_{\mathbb{Z}}$.)

5. Let G be a path-connected topological group, with identity element 1. Show that the fundamental group $\pi_1(G,1)$ is abelian.

6. Let $F_2 = \langle a, b\rangle$ be the free group of rank 2, let θ be the homomorphism $F_2 \to \mathbb{Z}_4$ given by $a \mapsto 1, b \mapsto 2$, and let $K = \ker\theta$. Find a minimal set of generators for K as a subgroup of F_2.

7. Show that $\mathbb{C}P^2$ is not homotopy equivalent to $S^2 \vee S^4$. Now, by considering the attaching map of the 4-cell in the standard cell decomposition of $\mathbb{C}P^2$, show that $\pi_3(S^2)$ is not trivial.

8. Prove that there is no closed 3-manifold which is homotopy-equivalent to the suspension $\Sigma\mathbb{R}P^2$.