Qualifying Exam: Mathematical Statistics 281AB

Fall 2025

Instructions: Provide rigorous and detailed mathematical arguments. Each statement should be supported by formal definitions, theorems, and proofs.

Total time allowed: **3 hours**. Total points: **36**.

Sufficiency and Rao–Blackwell improvement. A statistic $T = T(X)$ is *sufficient* for θ if there exists a factorization

$$f(x;\theta) \;=\; g_\theta\big(T(x)\big)\, h(x) \qquad \text{(Neyman–Fisher factorization).}$$

If $U = U(X)$ is any estimator with $\mathbb{E}_\theta|U| < \infty$ and T is sufficient, the *Rao–Blackwell* estimator

$$\delta^{\mathrm{RB}}(X) := \mathbb{E}_\theta[\,U \mid T\,]$$

is measurable in T and has risk no larger than U under every convex loss; in particular, for unbiased U, $\mathrm{Var}_\theta\big(\delta^{\mathrm{RB}}\big) \le \mathrm{Var}_\theta(U)$.

Problems

1. **Rao–Blackwell Improvement and UMP Testing (no completeness)** [14 pts]

   (i) **Rao–Blackwell theorem.** Prove that if T is sufficient for θ and U has finite mean, then for any convex loss L,

   $$\mathbb{E}_\theta\big[L\big(\delta^{\mathrm{RB}}(X) - \tau(\theta)\big)\big] \;\le\; \mathbb{E}_\theta\big[L\big(U(X) - \tau(\theta)\big)\big],$$

   where $\delta^{\mathrm{RB}}(X) = \mathbb{E}_\theta[U \mid T]$. Conclude that among unbiased estimators the variance cannot increase under Rao–Blackwellization. (Your proof should use sufficiency, conditional expectation, and Jensen's inequality.) [8 pts]

   (ii) **Poisson example: unbiased estimation of $e^{-2\theta}$ and RB improvement.** Let $X_1, \dots, X_n \overset{\text{i.i.d.}}{\sim} \text{Poisson}(\theta)$, $\theta > 0$.

   (a) Show that $U = \mathbf{1}\{X_1 = 0,\, X_2 = 0\}$ is unbiased for $e^{-2\theta}$. [2 pts]

   (b) Let $T = \sum_{i=1}^n X_i$. Compute $\delta^{\mathrm{RB}}(T) := \mathbb{E}[U \mid T]$ explicitly and show it equals

   $$\delta^{\mathrm{RB}}(T) = \Big(1 - \frac{2}{n}\Big)^T.$$

   Verify unbiasedness using the pgf of $T \sim \text{Poisson}(n\theta)$. [3 pts]

   (c) Using the law of total variance, show that $\mathrm{Var}_\theta\big(\delta^{\mathrm{RB}}(T)\big) \;\le\; \mathrm{Var}_\theta(U)$ with strict inequality for $\theta > 0$ and $n \ge 2$. (Optionally: derive the closed forms $\mathrm{Var}(\delta^{\mathrm{RB}}) = e^{n\theta((1-2/n)^2-1)} - e^{-4\theta}$ and $\mathrm{Var}(U) = e^{-2\theta}(1 - e^{-2\theta})$.) [2 pts]

2. **Nonregular Asymptotics for a Truncated Normal Cutoff** [22 pts]

   Let ϕ and Φ denote the standard normal pdf and cdf. For an unknown cutoff $\theta \in \mathbb{R}$, observe i.i.d. data

   $$X_1, \dots, X_n \overset{\text{i.i.d.}}{\sim} f(x;\theta) \;=\; \frac{\phi(x)}{1 - \Phi(\theta)}\,\mathbf{1}\{x \ge \theta\}, \qquad \text{(one–sided truncation of } N(0,1) \text{ at } \theta).$$

   Let θ_0 be the true cutoff and let

   $$\hat{\theta}_n \;=\; \arg\max_{\theta \in \mathbb{R}} \sum_{i=1}^n \log f(X_i;\theta)$$

   be the MLE.

   (i) **Form of the MLE.** Show that the likelihood is $L_n(\theta) \propto (1 - \Phi(\theta))^{-n}\, \mathbf{1}\{\theta \le X_{(1)}\}$ and conclude that

   $$\hat{\theta}_n \;=\; X_{(1)} := \min_{1 \le i \le n} X_i.$$

   (Argue that $\theta \mapsto (1 - \Phi(\theta))^{-n}$ is strictly increasing and the constraint $\theta \le X_{(1)}$ binds.) [6 pts]

   (ii) **Consistency.** Prove $\hat{\theta}_n \xrightarrow{P} \theta_0$. Compute the survival function under θ_0:

   $$\mathbb{P}_{\theta_0}(X \ge x) = \frac{1 - \Phi(x)}{1 - \Phi(\theta_0)}, \qquad x \ge \theta_0,$$

   and deduce $\mathbb{P}_{\theta_0}(\hat{\theta}_n > \theta_0 + \varepsilon) = \big[\frac{1 - \Phi(\theta_0 + \varepsilon)}{1 - \Phi(\theta_0)}\big]^n \to 0$ for any $\varepsilon > 0$, while $\mathbb{P}_{\theta_0}(\hat{\theta}_n < \theta_0) = 0$. [8 pts]

(iii) **Correct scaling and non-Gaussian limit.** Let

$$c_0 \;:=\; f_{\theta_0}(\theta_0^+) \;=\; \frac{\phi(\theta_0)}{1-\Phi(\theta_0)} \in (0,\infty),$$

the (right) density at the cutoff. Prove that

$$n\, c_0 \left(\hat{\theta}_n - \theta_0\right) \xrightarrow{d} \mathrm{Exp}(1).$$

(Hints: (a) for small $\delta > 0$, $\frac{1-\Phi(\theta_0+\delta)}{1-\Phi(\theta_0)} = 1 - c_0\,\delta + o(\delta)$; (b) use $\mathbb{P}_{\theta_0}(\hat{\theta}_n - \theta_0 > \delta) = \left[1 - c_0\delta + o(\delta)\right]^n$ with $\delta = t/n$.) [8 pts]

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Name: ______________________________

Total points: 20.

Let X and Y be independently distributed random variables with $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$. We aim to test $H_0 : \lambda \geq \mu$ versus $H_1 : \lambda < \mu$.

(a) [10 pts] Derive the uniformly most powerful unbiased (UMPU) test of size α given $X + Y = k$. Is it a randomized test or non-randomized test?

(b) [4 pts] Compute the power of the UMPU test in part (a) if $\lambda = 2, \mu = 4$, given $X+Y = k$.

(c) [6 pts] Justify how to derive the UMPU test of size α for testing $H_0 : \mu \leq \lambda$ versus $H_1 : \mu > \lambda$, where the independent samples are $X_1, \dots, X_n \sim \text{Poisson}(\lambda)$ and $Y_1, \dots, Y_m \sim \text{Poisson}(\mu)$.