

Name: _____

Question	Points	Score
1	5	
2	8	
3	7	
4	7	
5	5	
6	6	
7	7	
Total:	45	

1. Write your name on the front page of your exam.
2. Read each question carefully, and answer each question completely.
3. Write your solutions clearly in the exam sheet.
4. Show all of your work; no credit will be given for unsupported answers.
5. You may use the result of one part of the problem to prove a later part, even if you have not completed the earlier part.
6. You may use major theorems proved in class, but not when the whole point of the problem is to reproduce the proof of a theorem covered in class or in the textbook. When in doubt, ask us whether you are allowed to use a particular result.

1. (5 points) Suppose G is a group of order p^2q , where p and q are primes. Prove that G is not simple.

2. Suppose p is prime and G is a subgroup of the symmetric group S_p . Suppose p divides the order G . Let P be a Sylow p -subgroup of G , and H denote the subgroup of G generated by its Sylow p -subgroups.

(a) (2 points) Prove that $C_G(P) = P$, where $C_G(P)$ is the centralizer of P in G .

(b) (1 point) Prove that $N_G(P)/P$ injects into $\text{Aut}(P)$, where $N_G(P)$ is the normalizer of P in G .

(c) (1 point) Prove that $N_G(P)/N_H(P)$ is a cyclic group of order dividing $p - 1$.

(d) (3 points) Prove that

$$N_G(P)/N_H(P) \longrightarrow G/H, \quad xN_H(P) \longmapsto xH$$

is a surjective group homomorphism.

(e) (1 point) Prove that G/H is a cyclic group of order dividing $p - 1$.

3. Let F be a field. For $a \in M_n(F)$, let V_a be the $F[x]$ -module defined by $V_a := F^n$ with action $f(x) \cdot v := f(a)v$. For an $F[x]$ -module M , let $d(M)$ denote the minimum size of a generating set of M .
- (a) (4 points) Prove that the number of similarity classes of nilpotent matrices in $M_n(F)$ is equal to the number of partitions of n .

- (b) (3 points) Prove that if a is nilpotent, then $d(V_a) + \dim_F \operatorname{Im}(a) = n$.

4. Let A be a unital commutative ring
- (a) (5 points) Prove that every projective A -module is flat.

- (b) (2 points) Is every flat A -module projective? Justify your answer.

5. (5 points) Suppose $\phi: M \rightarrow M$ is a surjective A -module homomorphism and M is a Noetherian. Prove that ϕ is an isomorphism.

6. (6 points) Prove that $\mathbb{Q}[\cos(2\pi/n)]$ is a Galois extension of $\mathbb{Q}$ and find its Galois group.

7. Suppose F is a field, $f \in F[x]$ is irreducible and separable, $\deg f > 1$ and E is a splitting field of f over F . Suppose there exists $\alpha \in E$ such that $f(\alpha) = f(2\alpha + 1) = 0$.

(a) (4 points) Prove that the characteristic of F is an odd prime p .

(b) (3 points) Prove that the order of 2 in $(\mathbb{Z}/p\mathbb{Z})^\times$ divides $[E : F]$.

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Good Luck!