

Complex Analysis Qualifying Exam – Fall 2026

Name: _____

Student ID: _____

Instructions:

No books or notes are allowed. You may use without proofs results proved in Conway, Chapters I-XI. However, if using a homework problem, please make sure you reprove it. Present your solutions clearly, with appropriate detail.

You have 180 minutes to complete the test.

Notation: $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
Total		60

Problem 1. [10 points.]

Find all holomorphic functions $f : \mathbb{D} \rightarrow \mathbb{C}$ such that $f(0) = 1$, $f\left(\frac{1}{2}\right) = 3$, and $\operatorname{Re} f(z) > 0$ for all $z \in \mathbb{D}$.

Problem 2. [10 points; 5, 5.]

Let G be a region containing the closed unit disk $\overline{\mathbb{D}} \subset G$. Let $f : G \rightarrow \mathbb{C}$ be holomorphic such that

$$|f(z) - z^5| < 1$$

on the unit circle $|z| = 1$.

- (i) Show that there exists $\epsilon > 0$ such that, for every $|w| < \epsilon$, the equation $f(z) = w$ has precisely five roots in $\mathbb{D}$, counting multiplicity.

- (ii) Show that ϵ in part (i) may be chosen so that, for every $0 < |w| < \epsilon$, all five roots of $f(z) = w$ are simple.

Problem 3. [10 points; 5, 5.]

Let G be a region such that $0 \notin G$ and such that the winding number of every closed path γ in G with respect to 0 vanishes, i.e., $n(\gamma, 0) = 0$.

- (i) Show that there exists a holomorphic function g in G such that $e^{g(z)} = z$.

- (ii) Is it necessarily true that if h is holomorphic and never vanishes in G , then there is a holomorphic function f in G such that $e^{f(z)} = h(z)$? Prove it or give a counterexample.

Problem 4. [*10 points.*]

Let G be a region and $a \in G$. Let

$$\mathcal{F} = \{f : G \rightarrow \mathbb{C} \text{ holomorphic, } f(a) = -1, f \text{ omits every positive rational number}\}.$$

Show that $\mathcal{F}$ is a normal family.

Problem 5. [10 points; 5, 5.]

Let $S = \{z : |z| < 1, \operatorname{Im} z > 0\}$ denote the upper half unit disc.

- (i) Construct a biholomorphism between S and the quadrant $Q = \{z : \operatorname{Im} z > 0, \operatorname{Re} z > 0\}$.

(ii) Give an example of a harmonic function $h : S \rightarrow \mathbb{R}$ such that

$$\lim_{z \rightarrow a} h(z) = 0 \text{ for all } a \in \mathbb{R}, \ -1 < a < 1,$$

$$\lim_{z \rightarrow a} h(z) = \frac{\pi}{2} \text{ for all } |a| = 1, \ \text{Im } a > 0.$$

Problem 6. [10 points; 5, 5.]

(i) Show that there exists a holomorphic function $f : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$ such that

$$f\left(\frac{1}{n}\right) f\left(\frac{1}{n+1}\right) = n, \quad \forall n \in \mathbb{Z}_{>0}.$$

(ii) Show that any function f satisfying (i) must have an essential singularity at 0.