

Name _____ SID _____

Qualifying Exam in Numerical Analysis
Fall 2026

Time: 9:00 am – 12:00 noon, Tuesday, September 15, 2026

	Full Scores	Your Scores
# 1	25	
# 2	25	
# 3	25	
# 4	25	
# 5	25	
# 6	25	
# 7	25	
# 8	25	
Total	200	

Exam Rules and Instructions

- This is a three-hour, close-book, and close-note exam. No calculators, computers, tablets, and any other electronic devices are allowed. No cheatsheets are allowed. There are a total of 8 problems and a total 200 points.
- You must show all the steps for getting the answers. No credit will be given for unsupported answers.
- You may use, without proof, any results proved in lectures, but not as homework problems. If you use such a result, please cite it by its name (if it has one) or explain what it is concisely and verify the hypotheses. If the statement you are asked to prove is exactly a result in lecture or text, you still need to provide a proof instead of just citing the result.

Problem 1. (25 points) Let A be a real, $n \times n$, invertible matrix, and $b \neq 0$ be a real $n \times 1$ vector. Let $\|\cdot\|$ be a vector norm on $\mathbb{R}^n$ which induces a norm $\|\cdot\|$ on $\mathbb{R}^{n \times n}$. Assume now that $\delta A \in \mathbb{R}^{n \times n}$ is a matrix for which $\|\delta A\|/\|A\| < 1/\kappa_{\|\cdot\|}(A)$, where $\kappa_{\|\cdot\|}(A)$ is the condition number of A in the induced norm $\|\cdot\|$.

Suppose now that x and $x + \delta x$ are the solutions to the linear systems $Ax = b$, respectively, $(A + \delta A)(x + \delta x) = b$.

- (a) (1 point) Prove that, given the real numbers $a, b, c, d > 0$ such that $b > a$ and $d > c$, if $\frac{a}{b} \leq \frac{c}{d}$, then $\frac{a}{b-a} \leq \frac{c}{d-c}$.
- (b) (6 points) Prove that $A + \delta A$ is invertible, and hence $x + \delta x$ is well-defined.
- (c) (18 points) Prove that

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{\kappa_{\|\cdot\|}(A) \frac{\|\delta A\|}{\|A\|}}{1 - \kappa_{\|\cdot\|}(A) \frac{\|\delta A\|}{\|A\|}}.$$

Problem 2. (25 points) Let A be an $m \times n$ real matrix, and let B be the matrix obtained by rotating A 90 degrees clockwise on paper (this is not the same as the transpose!) For an example, see below:

$$A = \begin{bmatrix} 1 & -1 \\ 5 & 2 \\ 3 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 5 & 1 \\ -2 & 2 & -1 \end{bmatrix}.$$

Show that A and B have the same singular values. **You must prove this for the general case, not on the example shown above.**

Problem 3. (25 points)

- (a) (7 points) Suppose that A is an $n \times n$ complex matrix. Give an *efficient* algorithm for computing the eigenvalue and eigenvector of A closest to a given value μ via the inverse power iteration with shift. Assume that we take a fixed number k of iterations.

Assume that the eigenvalue closest to μ is unique, and explain why the algorithm will converge.

- (b) (8 points) What is the complexity of the algorithm above, as a function of n and k ?
- (c) (8 points) Assume now that the matrix is positive definite and tridiagonal. Explain how to exploit the matrix structure to make the algorithm as efficient as possible, and what the new complexity will be.

Problem 4. (25 points) Let $n \geq 2$ be an integer, $f \in C^1(\mathbb{R}^n)$, and $x_0 \in \mathbb{R}^n$. The steepest-descent method for minimizing f is given by $x_{k+1} = x_k - \lambda_k \nabla f(x_k)$ ($k = 0, 1, \dots$), where λ_k is determined by

$$\lambda_k = \arg \min_{\lambda \geq 0} f(x_k - \lambda \nabla f(x_k)).$$

Let $k \geq 1$ and assume $\lambda_k > 0$. Prove that $f(x_{k+1}) \leq f(x_k)$ and $\nabla f(x_k) \cdot \nabla f(x_{k+1}) = 0$.

Problem 5. (25 points) (a) Let $-\infty < a < b < \infty$, $f \in C([a, b]) \setminus \mathcal{P}_n$, and $p_n \in \mathcal{P}_n$ for some integer $n \geq 0$, where $\mathcal{P}_n$ denotes the class of real polynomials of degree less than or equal to n . Prove that p_n is the least-squares approximation of f on $[a, b]$ in $\mathcal{P}_n$ (i.e., $\|f - p_n\|_{L^2(a,b)} \leq \|f - q_n\|_{L^2(a,b)}$ for all $q_n \in \mathcal{P}_n$) if and only if $\langle f - p_n, r_n \rangle = 0$ for all $r_n \in \mathcal{P}_n$, where for any $u, v \in L^2(a, b)$

$$\langle u, v \rangle = \int_a^b uv \, dx \quad \text{and} \quad \|u\|_{L^2(a,b)} = \sqrt{\langle u, u \rangle}.$$

(b) Let $\tilde{\mathcal{P}}_n$ denote the class of real polynomials of degree n with the leading coefficient 1. Let $-\infty < a < b < \infty$ and $\tilde{Q}_0, \tilde{Q}_1, \dots$ be orthogonal polynomials in $L^2(a, b)$ with $\tilde{Q}_n \in \tilde{\mathcal{P}}_n$ for each $n \geq 0$. Fix $n \geq 1$. Prove that $\|\tilde{Q}_n\|_{L^2(a,b)} \leq \|\tilde{q}_n\|_{L^2(a,b)}$ for all $\tilde{q}_n \in \tilde{\mathcal{P}}_n$.

Hint: For Part (b), you can use the result of Part (a) (even if you cannot prove that) with $f(x) = x^n$ and $\mathcal{P}_n$ replaced by $\mathcal{P}_{n-1}$.

Problem 6. (25 points) Let m and n be integers with $0 \leq m \leq n$ and $x_0, \dots, x_n$ be distinct points in a finite interval (a, b) ($a < b$). Let $l_0, \dots, l_n \in \mathcal{P}_n$ and $L_n : C([a, b]) \rightarrow \mathcal{P}_n$ be the Lagrange basis polynomials and interpolation operator, respectively, associated with $x_0, \dots, x_n$. (The notation $\mathcal{P}_n$ is the same as in Problem 5.) Define $K_m : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$K_m(x, t) = \frac{1}{m!} \left[(x - t)_+^m - \sum_{k=0}^n (x_k - t)_+^m l_k(x) \right],$$

where $c_+ = \max(c, 0)$, $c_+^m = (c_+)^m$ if $m \geq 1$, and $c_+^0 = 1$ if $c \geq 0$ and $c_+^0 = 0$ if $c < 0$. Let $f \in C^{m+1}([a, b])$. By the Taylor expansion, $f(x) = Q_m(x) + R_m(x)$ with $Q_m \in \mathcal{P}_m$ and

$$R_m(x) = \frac{1}{m!} \int_a^x (x - t)^m f^{(m+1)}(t) dt.$$

Prove that $f - L_n f = R_m - L_n R_m$ and that

$$f(x) - (L_n f)(x) = \int_a^b K_m(x, t) f^{(m+1)}(t) dt \quad \forall x \in [a, b].$$

Problem 7. (25 points) Let $\gamma \in \mathbb{R}$ and consider the two-stage Runge–Kutta method with the Butcher tableau

$$\begin{array}{c|cc} \gamma & \gamma & 0 \\ 1 & 1-\gamma & \gamma \\ \hline & 1-\gamma & \gamma \end{array}$$

applied to the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$, with stepsize h .

- (a) (8 points) Apply the method to the model problem

$$y' = \lambda y, \quad t \geq 0, \quad y(0) = 1,$$

and compute the stability function $R(z)$, $z = h\lambda$, which satisfies $y_{n+1} = R(z)y_n$. Show all the steps, and simplify the result.

- (b) (9 points) A method of order p satisfies $R(z) - e^z = O(z^{p+1})$ as $z \rightarrow 0$. Use this to find every γ for which the order can be at least 2, and prove that no member of the family has order 3. Explain why this argument gives a necessary condition only, and state what remains to be verified.
- (c) (8 points) Find every $\gamma \in \mathbb{R}$ for which the method is A-stable, that is, for which the linear stability domain D contains $\mathbb{C}^- = \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$. Show that the values of γ found in (b) are exactly the endpoints of this set, and that for those two methods $R(z) \rightarrow 0$ as $|z| \rightarrow \infty$.

Hint. Lemma 4.3 may be used without proof: let R be a rational function that is not constant; then $|R(z)| < 1$ for all $z \in \mathbb{C}^-$ if and only if every pole of R has positive real part and $|R(it)| \leq 1$ for all $t \in \mathbb{R}$.

Problem 8. (25 points) Consider the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$, on the uniform grid $t_n = t_0 + nh$, and write $f_n = f(t_n, y_n)$. The two-step backward differentiation formula is constructed as follows: let $p \in \mathcal{P}_2$ interpolate the values y_n, y_{n+1}, y_{n+2} at the nodes t_n, t_{n+1}, t_{n+2} , and impose the differential equation on p at the single point t_{n+2} , that is, $p'(t_{n+2}) = f_{n+2}$.

- (a) (10 points) Derive the method from this construction. Show all the steps, including the Lagrange basis polynomials and their derivatives. Determine the order p of the method and the error constant c .
- (b) (7 points) Determine whether the method is convergent. Compute the zeros of ρ explicitly, and justify the answer with the root condition and the Dahlquist equivalence theorem.
- (c) (8 points) The three-step backward differentiation formula is

$$y_{n+3} - \frac{18}{11}y_{n+2} + \frac{9}{11}y_{n+1} - \frac{2}{11}y_n = \frac{6}{11}hf_{n+3}.$$

Determine whether this method is convergent. Compute the zeros of its polynomial ρ explicitly, and justify the answer as in part (b).