

QUALIFYING EXAM: REAL ANALYSIS

September 9, 2026 (180 minutes)

No books, notes, or electronic devices are permitted during this exam. Please turn off all cell phones and put them away.

You must show your work to receive credit. Present your solutions clearly, and indicate which result you are using at each step.

You may use without proof any result (theorem, proposition, lemma, corollary) contained in Chapters 1-8 of *Gerald B. Folland, Real Analysis*. Be sure to explicitly check that all necessary assumptions are satisfied.

Name (print): _____

Student ID number: _____

1. [15 pts] Let $X = (0, 1]$, and let $\mathcal{A}$ be the algebra consisting of finite disjoint unions of intervals of the form $(a, b] \subseteq (0, 1]$. Define

$$\mu_0\left(\bigsqcup_{k=1}^m (a_k, b_k]\right) = \sum_{k=1}^m (\sqrt{b_k} - \sqrt{a_k}),$$

where the union is disjoint.

- (1) Show that μ_0 is a premeasure on $\mathcal{A}$, and explain why it has a unique extension to a finite measure μ on $\mathcal{B}((0, 1])$, the Borel sigma algebra on $(0, 1]$.
- (2) Find a density w such that

$$\mu(E) = \int_E w(x) dx$$

for every Borel set $E \subseteq (0, 1]$.

- (3) Define another finite measure by

$$\nu(E) = \int_E x d\mu(x).$$

Decide whether $\mu \ll \nu$. If so, determine $\frac{d\mu}{d\nu}$.

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2. [20 pts] Throughout, $|A|$ denotes Lebesgue measure in $\mathbb{R}^d$. Let $E \subset \mathbb{R}^d$ be measurable, let $K \subset E$ be compact, and for $\rho > 0$ define

$$K_\rho := \{z \in \mathbb{R}^d : \text{dist}(z, K) < \rho\}.$$

For $0 < \alpha < 1$, set

$$A_{\alpha, \rho}(K, E) := \left\{ x \in K : \text{there exists } 0 < r < \rho \text{ such that } \frac{|B_r(x) \setminus E|}{|B_r(x)|} > \alpha \right\}.$$

A measurable set $S \subset \mathbb{R}^d$ is called *uniformly porous at arbitrarily small scales* if there exists $c \in (0, 1)$ such that, for every $x \in S$ and every $\rho > 0$, there are $0 < r < \rho$ and $y \in \mathbb{R}^d$ satisfying

$$B_{cr}(y) \subset B_r(x) \setminus S.$$

(1) For each fixed $r > 0$, prove that

$$x \mapsto |B_r(x) \setminus E|$$

is continuous. Deduce that $A_{\alpha, \rho}(K, E)$ is relatively open in K , and hence measurable.

(2) Prove the estimate

$$|A_{\alpha, \rho}(K, E)| \leq \frac{5^d}{\alpha} |K_\rho \setminus E|.$$

(Hint: Use the $5r$ -Vitali covering lemma.)

(3) Suppose that S is uniformly porous at arbitrarily small scales with porosity constant c . Let $K \subset S$ be compact and choose $0 < \alpha < c^d$. Prove that, for every $\rho > 0$,

$$A_{\alpha, \rho}(K, S) = K.$$

Deduce that

$$|K| \leq \frac{5^d}{\alpha} |K_\rho \setminus S| \quad \text{for every } \rho > 0.$$

(4) Prove that every measurable set $S \subset \mathbb{R}^d$ that is uniformly porous at arbitrarily small scales has Lebesgue measure zero.

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3. [15 pts] Assume that $1 < p < \infty$. For $m \geq 0$, let

$$I_{m,k} = [k2^{-m}, (k+1)2^{-m}), \quad k = 0, \dots, 2^m - 1,$$

where the last interval is understood to contain the endpoint 1, and set

$$V_m = \text{span}\{1_{I_{m,k}} : 0 \leq k < 2^m\}.$$

For $f \in L^p([0, 1])$, define the dyadic averaging operator

$$P_m f = \sum_{k=0}^{2^m-1} \left(\frac{1}{\lambda(I_{m,k})} \int_{I_{m,k}} f d\lambda \right) 1_{I_{m,k}}.$$

- (1) Use Hölder's inequality to prove that P_m is a bounded linear projection from $L^p([0, 1])$ onto V_m , and show that $\|P_m\|_{L^p \rightarrow L^p} = 1$.
- (2) Prove that $P_m f \rightarrow f$ in $L^p([0, 1])$ for every $f \in L^p([0, 1])$. (Hint: first treat bounded functions using the Lebesgue differentiation theorem and the dominated convergence theorem, and then pass to general L^p functions by truncation)
- (3) For the sequence $f_n = n^{1/p} 1_{(0, 1/n)}$, prove that, for each fixed m ,

$$\|P_m f_n\|_p \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

although $\|f_n\|_p = 1$. Evaluate the two iterated limits

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \|P_m f_n\|_p \quad \text{and} \quad \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \|P_m f_n\|_p.$$

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4. [20 pts] Let X be a locally compact Hausdorff space and $M(X) = C_0(X)^*$ the space of all complex Radon measures on X . Recall that the norm of $\nu \in M(X)$ is $\|\nu\| = |\nu|(X)$, where $|\nu|$ is the total variation of ν . Let $\mu \in M(X)$ be a positive measure and define

$$\text{Abs}(\mu) = \{\nu \in M(X) : \nu \ll \mu\}.$$

- (1) Prove that $\text{Abs}(\mu)$ is a closed subspace in the norm topology.
- (2) Using Radon-Nikodym derivatives, define $T : \text{Abs}(\mu) \rightarrow L^1(X, \mu)$ by $T(\nu) = \frac{d\nu}{d\mu}$. Prove that T is an isometry.
- (3) Prove that T is surjective.
- (4) Let $\nu \in \text{Abs}(\mu)$ and let (ν_n) be a sequence in $\text{Abs}(\mu)$. Prove that if $(T(\nu_n))$ converges to $T(\nu)$ weakly in $L^1(X, \mu)$ then (ν_n) converges to ν weak* in $M(X)$.

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5. [15 pts] Let $(X, \mathcal{M}, \mu)$ be a measure space, let $1 < p < \infty$ and let $f_n \in L^p(X, \mu)$ for $n \in \mathbb{N}$. Assume that $\sup \|f_n\|_p < \infty$ and that $f_n \rightarrow f$ in $L^1(X, \mu)$ for some $f \in L^1(X, \mu)$.

- (1) Prove that $f \in L^p(X, \mu)$ and that $f_n \rightarrow f$ in $L^q(X, \mu)$ for every $q \in [1, p)$.
- (2) Does the same hold when $q = p$? Justify your answer.

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6. [15 pts] Let X and Y be Banach spaces and $T \in L(X, Y)$. Define $T^* : Y^* \rightarrow X^*$ by $T^*(g)x = g(Tx)$. Prove that if X is reflexive and T is injective then $T^*(Y^*)$ is dense in X^* .

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7. [25 pts] Let $a > 0$ and $f \in C_c^\infty(\mathbb{R}^n)$, and let $\Delta := \sum_{j=1}^n (\partial_j)^2$ be the Laplacian operator on $\mathbb{R}^n$. Prove that the partial differential equation $\Delta u - au = f$ on $\mathbb{R}^n$ has exactly one classical solution $u \in L^2(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ such that also $\partial^\alpha u \in L^1(\mathbb{R}^n)$ for each multiindex α with $|\alpha| \in \{1, 2\}$, and find it in terms of a, f , and the Fourier transform. Here “classical solution” means that all terms in the equation are continuous functions and the equality holds pointwise everywhere. (*Hint:* First find the formula for u assuming that u has the given properties. Then show that the u given by the formula has those properties; to get $\partial^\alpha u \in L^1(\mathbb{R}^n)$, show that $\partial^\alpha u$ decays fast enough.)

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