

Fall 2026 Applied Algebra Qualifying Exam

Math 202 A/B/C

Instructions.

1. There are ten problems on this exam.
2. Give complete proofs and justify all assertions.
3. Throughout this exam, the term *Hilbert space* refers to a finite-dimensional complex vector space V equipped with a scalar product linear in the second argument.
4. The term *algebra* refers to a finite-dimensional complex vector space $\mathcal{A}$ equipped with an associative, bilinear, unital multiplication and an antilinear, antimultiplicative, involutive conjugation.
5. All algebra homomorphisms respect multiplication, conjugation, and units.

Problem 1.

Let V be a Hilbert space and let $P \in \text{End}(V)$ be a nonzero idempotent, so that $P^2 = P$. Prove that $\|P\| \geq 1$, and that $\|P\| = 1$ if and only if $P = P^*$.

Problem 2.

Let A be an $m \times n$ matrix whose columns are one-hot binary vectors: each column contains exactly one entry equal to 1 and all other entries are equal to 0. Let r_i denote the sum of the entries in row i of A . Determine the nonzero singular values of A , and give explicit orthonormal bases for each of the corresponding left and right singular spaces.

Problem 3.

Let (V, φ) and (W, ψ) be finite-dimensional unitary representations of a group G . Let $A \in \text{Hom}_G(V, W)$ be a G -equivariant linear map with operator norm $\sigma_1 > 0$. Prove that

$$V_1 = \{v \in V : \|Av\| = \sigma_1 \|v\|\}$$

is a subrepresentation of (V, φ) .

Problem 4.

Let V be a Hilbert space and let $\mathcal{A} \subseteq \text{End}(V)$ be a unital subalgebra closed under adjoints. Prove that $\mathcal{A}$ is commutative if and only if every orthogonal projection belonging to $\mathcal{A}$ commutes with every element of $\mathcal{A}$.

Problem 5.

Let V be a Hilbert space of dimension $n \geq 1$. Prove that every state σ on $\text{End}(V)$ has a unique expression

$$\sigma(X) = \text{Tr}(DX),$$

where $D \in \text{End}(V)$ is positive semidefinite and $\text{Tr } D = 1$. Conversely, show that every such D defines a state. Characterize which of these states are faithful and which are tracial.

Problem 6.

Let $T \in \text{End}(V)$ be normal, with distinct eigenvalues $t_1, \dots, t_r$ and eigenspaces $V_1, \dots, V_r$. Write

$$Z(T) = \{X \in \text{End}(V) : XT = TX\}.$$

Determine $Z(T)$ and its centralizer in $\text{End}(V)$ as $*$ -algebras. Prove that the latter is the polynomial algebra $\mathbb{C}[T]$, and determine exactly when $\mathbb{C}[T]$ is a maximal commutative subalgebra of $\text{End}(V)$.

Problem 7.

Let $G = (S, T)$ be the graph whose vertex set S consists of all bitstrings of length 3, with $\{s_1, s_2\} \in T$ an edge if and only if s_1 and s_2 differ by a single bit. Let $A \in \text{End } \mathcal{F}(S)$ be the adjacency operator of G . Find the eigenvalues of A , and give an orthonormal basis for each eigenspace.

Problem 8.

Let G be a nontrivial finite group of odd order. Show that the convolution algebra $\mathcal{C}(G)$ contains a subalgebra $\mathcal{B}$ not isomorphic to $\mathcal{C}(H)$ for any subgroup $H \leq G$.

Problem 9.

Let $n \geq 2$. Let $S_n = \text{Aut}\{1, \dots, n\}$ be the symmetric group, and consider the unitary representation (V, φ) of S_n in which

$$V = \bigoplus_{i=1}^n \mathbb{C}|i\rangle, \quad \varphi(\pi)|i\rangle = |\pi(i)\rangle.$$

Decompose (V, φ) into irreducible unitary representations of S_n , with proof. Write out the character table of S_3 .

Problem 10.

Let G be a finite group and let H be a subgroup of G . Let $\Lambda(G)$ be a set parameterizing irreducible unitary representations V^λ of G , and let $\Lambda(H)$ be a set parameterizing irreducible unitary representations W^μ of H . For $\lambda \in \Lambda(G)$ and $\mu \in \Lambda(H)$, let $m_{\lambda\mu}$ be the multiplicity of W^μ in the restriction of V^λ to H . Omitting terms with $m_{\lambda\mu} = 0$, show that the centralizer of $\mathcal{C}(H)$ in $\mathcal{C}(G)$ is isomorphic to

$$\bigoplus_{\lambda \in \Lambda(G)} \bigoplus_{\mu \in \Lambda(H)} \text{End}(\mathbb{C}^{m_{\lambda\mu}}).$$