

NUMERICAL ANALYSIS QUALIFYING EXAMINATION
INSTRUCTORS: IOANA DUMITRIU, BO LI, MELVIN LEOK
SPRING 2026

NAME: _____

Part I: Math 270A

Problem	Points Possible	Points Earned
1	25	
2	25	
3	25	
Total	75	

1. (25 points) Let x, y, z be real numbers, and consider computing $\alpha = x \cdot y - z$ in floating point arithmetic with unit ε .

- (a) (5 points) Write out the expression of the computed $\hat{\alpha}$.
- (b) (7 points) Consider the relative forward error $\frac{|\alpha - \hat{\alpha}|}{|\alpha|}$ and prove a bound on it of the type

$$\frac{|\alpha - \hat{\alpha}|}{|\alpha|} \leq |f(x, y, z)| \varepsilon + O(\varepsilon^2) ,$$

with an explicit $f(x, y, z)$.

- (c) (7 points) Prove backward stability for the computation, namely, show that there exist $\tilde{x}, \tilde{y}, \tilde{z}$ such that, in exact arithmetic, $\tilde{x} \cdot \tilde{y} - \tilde{z} = \hat{\alpha}$ and

$$\max \left\{ \frac{|\tilde{x} - x|}{|x|}, \frac{|\tilde{y} - y|}{|y|}, \frac{|\tilde{z} - z|}{|z|} \right\} \leq C \varepsilon ,$$

computing C explicitly.

- (d) (6 points) Explain why forward stability of this operation is problematic, although the backward stability is not.

2. (25 points) Let A be a full-rank, real $m \times n$ matrix.
- (a) (9 points) Prove that it has a factorization $A = QL$, with Q an $m \times n$ matrix with orthonormal columns, and L a lower triangular matrix with positive diagonal entries. What does this factorization represent?
 - (b) (8 points) Sketch an algorithm for how one would compute this factorization, and explain its complexity. (Any algorithm that works exactly in exact arithmetic is good here.)
 - (c) (8 points) Given a black box algorithm `bb_q1` for computing the QL factorization of a matrix, explain how, given any matrix A of full rank, we can use `bb_q1` to obtain the QR factorization of A . Your algorithm for doing so must include **a single call** to `bb_q1` and **perform no additional flops**.

3. (25 points) Consider the following $n \times n$ “arrow” matrix

$$A = \begin{bmatrix} a_{11} & & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & a_{33} & \cdots & a_{3n} \\ & & & \ddots & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

with nonzero entries only on the diagonal, last column, and last row. Assume it is invertible and strictly both row- and column-diagonally dominant ($|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ and $|a_{ii}| > \sum_{j \neq i} |a_{ji}|$, for all i).

- (a) (9 points) Write down a simple algorithm for solving the system $Ax = b$ for a $n \times 1$ vector b , through the most efficient and stable direct method you know, and compute its complexity, as close to exact as you can (taking into account the special shape of A).
- (b) (9 points) Consider now the Jacobi iteration algorithm for solving the same system. Write it down as efficiently as possible (again, taking into account the special shape of A), prove why it works, and state its complexity, given some tolerance δ .
- (c) (7 points) Explain which of these two methods you would use, and why.

Part II: Math 270B

Problem	Points Possible	Points Earned
4	25	
5	25	
6	25	
Total	75	

4. (25 points) Newton's iteration for solving $f(x) = 0$ on $[a, b]$ ($a, b \in \mathbb{R}$ and $a < b$) is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad (n = 0, 1, \dots).$$

Assume $f(x) = 0$ has a unique solution x^* on $[a, b]$ with $a < x^* < b$. Assume $f \in C^2([a, b])$ and $f'(x) \geq m$ and $0 < f''(x) \leq M$ for all $x \in [a, b]$, where m and M are two positive constants. Let $x_0 \in (x^*, b)$. Prove $x^* < x_1 < x_0$ and

$$|x_1 - x^*| \leq \frac{M}{2m} |x_0 - x^*|^2.$$

5. (a) (10 points) Let $x_0, x_1, \dots, x_n \in [a, b]$ ($a, b \in \mathbb{R}$ and $a < b$) be distinct. Define for each k ($0 \leq k \leq n$)

$$l_k(x) = \prod_{0 \leq j \leq n, j \neq k} \frac{x - x_j}{x_k - x_j} \quad \forall x \in [a, b].$$

Let $m \in \{0, 1, \dots, n\}$. Prove $\sum_{k=0}^n x_k^m l_k(x) = x^m$ for all $x \in [a, b]$.

- (b) (15 points) Let $p \in \mathcal{P}_{20}$ ($\mathcal{P}_n$ denotes the class of real polynomials of degree less than or equal to n with $n \geq 0$ an integer) interpolate $f(x) = 1 + x + x^2 + x^3$ at 21 points: $x_i = 2^{-i}$ ($i = 0, 1, \dots, 20$). Find out the explicit expression of p and justify your answer.

6. (25 points) Let $f \in C([a, b])$ ($a, b \in \mathbb{R}$ and $a < b$). Consider the sequence of Gaussian formulas and their errors

$$\int_a^b f(x) dx \approx \sum_{j=1}^n A_j^{(n)} f(x_j^{(n)}) \quad \text{and} \quad e_n(f) = \int_a^b f(x) dx - \sum_{j=1}^n A_j^{(n)} f(x_j^{(n)}) \quad (n = 1, 2, \dots).$$

Prove for each n that $A_j^{(n)} > 0$ ($j = 1, \dots, n$), $\sum_{j=1}^n A_j^{(n)} = b - a$, and

$$|e_n(f)| \leq 2(b-a) \min_{q \in \mathcal{P}_{2n-1}} \|f - q\|_{C([a,b])}.$$

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Part III: Math 270C

Problem	Points Possible	Points Earned
7	25	
8	25	
Total	50	

7. (a) (9 points) Derive a 3-stage collocation Runge–Kutta method with collocation points $c_1 = 0$, $c_2 = \frac{1}{2}$, $c_3 = 1$, and write down the resulting method in the form of a Butcher tableau.
- (b) (8 points) Compute the stability function $R(h\lambda)$ for the collocation Runge–Kutta method you derived above, which satisfies the equation

$$y_{n+1} = R(h\lambda)y_n$$

when the Runge–Kutta method is applied to the model problem,

$$\begin{cases} y'(t) = \lambda y(t), & t > 0, \\ y(0) = 1. \end{cases}$$

- (c) (8 points) By using Lemma 4.3, which states that $|R(z)| < 1$ for all $\mathbb{C}^-$ iff all the poles of $R(z)$ have positive real parts and $|R(it)| \leq 1$ for all $t \in \mathbb{R}$, prove that the Lobatto IIIA method you derived above is A-stable.

8. (25 points) Consider the following α -parameterized family of linear multistep methods,

$$u_{n+1} - \alpha u_n = h \left[(1 - \alpha) f_{n+1} + \frac{3\alpha}{2} f_n - \frac{\alpha}{2} f_{n-1} \right].$$

Determine how the stability (root condition), order of accuracy, and convergence (Dahlquist equivalence theorem) properties of this family of methods changes as we allow $\alpha \in \mathbb{R}$ to vary.