Representation Theory of $\mathrm{GL}_2(\mathbb{F}_q)$

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1 Introduction

The aim of this thesis is to classify the irreducible complex representations of $GL_2(\mathbb{F}_q)$, where $\mathbb{F}_q$ is a finite field of characteristic p with q elements. In the first section, we lay the groundwork by introducing some of the basic results in representation theory. Then, we introduce the Cartan and Borel subgroups of $GL_2(\mathbb{F}_q)$. Next, we find the conjugacy classes of $GL_2(\mathbb{F}_q)$, which naturally fit into four types based on the Jordan Canonical Form. Finally, we find four types of irreducible complex representations of $GL_2(\mathbb{F}_q)$ by inducing from the Cartan and Borel subgroups. We show that the number of irreducible representations of each type are as many as the conjugacy classes in each type. We will demonstrate that these four types of representation, the 1-Dimensional, Steinberg, Principal Series, and Cuspidal representations, together contain all irreducible representations of $GL_2(\mathbb{F}_q)$.

2 Definitions and Notation

In the setting of this paper, we denote by G a finite group and denote by V a finite-dimensional $\mathbb{C}$-vector space. $GL(V)$ denotes the group of automorphisms of V.

Definition 2.1. A *representation* of G in V is a group homomorphism $\rho : G \to GL(V)$.

Remark 2.2. If ρ is a linear representation of G, and $s \in G$, we sometimes denote $\rho_s := \rho(s)$.
We commonly call the vector space V a *representation* of G when ρ is clear from the context.

Example 2.3. Suppose ρ is a representation of G in $GL(\mathbb{C}) \simeq \mathbb{C}^\times$. If ρ is the representation such that $\rho(s) = 1$ for all $s \in G$, ρ is called the *trivial representation.*

Example 2.4. Suppose that V is of dimension $|G|$, and V has a basis indexed by the elements of G, $\{e_t\}_{t\in G}$. Let ρ be the representation of G in V defined by $\rho_s(e_t) = e_{st}$ for all $s, t \in G$. ρ is called the *regular representation* of G.

Example 2.5. Let $\rho^V : G \to GL(V)$ be a representation of G. Then we can define a representation of G in the dual vector space V^* =Hom$(V, \mathbb{C})$ called the *dual representation* of V. ρ^* is defined by $\rho^*(g) := \rho^V(g^{-1})^T$ for all $g \in G$.

Remark 2.6. For all $v^* \in V^*, v \in V, g \in G$

$$\langle \rho_g^*(v^*), \rho_g^V(v) \rangle = (v^*)^T \rho^V(g^{-1}) \rho^V(g) v = (v^*)^T v = \langle v^*, v \rangle$$

. We will see how this is useful shortly.

Remark 2.7. Let V and W be representations of G. Then $V \oplus W$ and $V \otimes W$ are also representations of G; the former via $\rho_s^{V \oplus W}(v \oplus w) = \rho_s^V(v) + \rho_s^W(w)$ and the latter via $\rho_s^{V \otimes W}(v \otimes w) = \rho_s^V(v) \otimes \rho_s^W(w)$.

Definition 2.8. If $\rho^1 : G \to GL(V)$ and $\rho^2 : G \to GL(W)$ are representations, then a linear map $\varphi : V \to W$ is a $G-$ *linear* map from the representation V to W if $\varphi(\rho_g^1(v)) = \rho_g^2(\varphi(v))$ for all $v \in V$ and $g \in G$. In other words, the diagram

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho_g^1"] & W \arrow[d, "\rho_g^2"] \\ V \arrow[r, "\varphi"] & W \end{tikzcd}$$

commutes. V and W are said to be isomorphic as representations if there exists a bijective G-linear map φ from V to W.

Proposition 2.9. Suppose that $\rho^V : G \to GL(V)$ and $\rho^W : G \to GL(W)$ are representations of G. We define a representation $\rho : G \to GL(\text{Hom}(V, W))$ by the identification $\text{Hom}(V, W) \simeq V^* \otimes W$. Then, for any map $\varphi \in \text{Hom}(V, W)$, $\rho_g(\varphi)(\rho_g^V(v)) = \rho_g^W(\varphi(v))$ for all $g \in G$ and $v \in V$. In other words, the diagram

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho_g^V"] & W \arrow[d, "\rho_g^W"] \\ V \arrow[r, "\rho_g(\varphi)"] & W \end{tikzcd}$$

commutes.

Proof. We prove the result holds for any simple tensor, from which it follows that it holds in general. Suppose $\varphi = v^* \otimes w \in V^* \otimes W$. Then, $\rho_g(\varphi)(\rho_g^V(v)) = \langle \rho_g^*(v^*), \rho_g^V(v)\rangle \rho_g^W(w) = \langle v^*, v\rangle \rho_g^W(w)$.
On the other hand, $\rho_g^W(\varphi(v)) = \rho_g^W(\langle v^*, v\rangle w) = \langle v^*, v\rangle \rho_g^W(w)$. Therefore, $\rho_g(\varphi)(\rho_g^V(v)) = \rho_g^W(\varphi(v))$

Corollary 2.10. The G-linear maps between representations V and W are precisely the maps fixed by the representation of $\mathrm{Hom}(V, W)$.

Proof. Suppose $\varphi \in \mathrm{Hom}(V, W)$ is fixed by ρ_g for all $g \in G$. Then, we have that $\varphi(\rho_g^V(v)) = \rho_g^W(\varphi(v))$ for all $v \in V$ and $g \in G$ by the proposition, so φ is G-linear.
On the other hand, suppose that φ is a G-linear map from V to W. Then, $\varphi(\rho_g^V(v)) = \rho_g^W(\varphi(v))$ and by the proposition $\varphi(\rho_g^V(v)) = \rho_g(\varphi)(\rho_g^V(v))$ for all $v \in V$ and $g \in G$. Therefore $\rho_g(\varphi) = \varphi$ for all $g \in G$.

Definition 2.11. Suppose that $\rho : G \to GL(V)$ is a representation, and W is a vector subspace of V. Then, if $\rho_s(W) \subseteq W$ for all $s \in G$, we say that W is a *subrepresentation* of V.

It follows from the definition that the following statements are equivalent for a given linear representation $\rho : G \to GL(V)$:

(**1**) The subspace $W \subseteq V$ is a subrepresentation of V.

(**2**) The restriction of ρ to the subspace W is a well-defined linear representation of G. In this case the restriction $\rho^W : G \to GL(W)$ is given by $\rho_s^W(w) = \rho(w)$ for all $w \in W$ and $s \in G$.

Definition 2.11. We call a representation V *irreducible* if V has no proper nontrivial subrepresentation.

Theorem 2.12 Let $\rho : G \to GL(V)$ be a linear representation of G in a finite dimensional $\mathbb{C}$-vector space V and let W be a subrepresentation of V. Then, there is a complement W^0 of W in V that is also a subrepresentation of V.

Proof. Let $p : V \to W$ be the linear projection of V onto W. Define

$$p^0 := \frac{1}{|G|} \sum_{t \in G} \rho_t \cdot p \cdot \rho_t^{-1}$$

p^0 is also a linear projection of V onto W.
Therefore $W^0 :=$ Ker (p^0) is a complement of W in V. Moreover,

$$\rho_s \circ p^0 \rho_s^{-1} = \frac{1}{|G|} \sum_{t \in G} \rho_s \rho_t \cdot p \cdot \rho_t^{-1} \rho_s^{-1} = \frac{1}{|G|} \sum_{t \in G} \rho_{st} \cdot p \cdot \rho_{(st)^{-1}} = p^0$$

For any $x \in W^0$ and $s \in G$, $p^0 \circ \rho_s(x) = \rho_s \circ p^0(x) = 0$. Therefore $\rho_s(W^0) \subseteq W^0$ for all $s \in G$.

Corollary 2.13 (**Semisimplicity**). Any linear representation V can be written as a direct sum of irreducible representations.

Proof. We can prove this by induction on the dimension of V. If V is irreducible, we are done. Otherwise, by the theorem, we can write the representation V as the direct sum of subrepresentations W and W^0, which are of a strictly lower dimension than V. By the induction hypothesis, there are irreducible representations $\{W_1, ..., W_n\}$ and $\{W_1^0, ..., W_m^0\}$ such that $W = W_1 \oplus ... \oplus W_n$ and $W^0 = W_1^0 \oplus ... \oplus W_m^0$. Therefore, $V = W_1 \oplus ... \oplus W_n \oplus W_1^0 \oplus ... \oplus W_m^0$.

Theorem 2.14 (**Schur's Lemma**) Let $\rho^1 : G \to GL(V_1)$ and $\rho^2 : G \to GL(V_2)$ be two irreducible representations of G, and let f be a G-linear map from V_1 to V_2 satisfying $\rho_s^2 \circ f = f \circ \rho_s^1$ for all $s \in G$.
(**1**) If V_1 and V_2 are not isomorphic representations, $f = 0$.

(2) If V_1 and V_2 are isomorphic representations, then $f = \lambda I$ for some $\lambda \in \mathbb{C}$.

Proof. (**1**) Suppose V and W are not isomorphic representations, but $f \neq 0$. For all $v \in \operatorname{Ker}(f)$ and $s \in G$, we have that $\rho_s^2 \circ f(v) = 0$, implying that $f \circ \rho_s^1(v) = 0$. Therefore, $\operatorname{Ker}(f)$ is a subrepresentation of V_1, and must be either V_1 or 0. Since $f \neq 0$ by assumption, $\operatorname{Ker}(f) = 0$. On the other hand, for all $v \in \operatorname{Im}(f)$ and $s \in G$, we have that $f \circ \rho_s^1(v) \in \operatorname{Im}(f)$, implying that $\rho_s^2 \circ f(v) \in \operatorname{Im}(f)$. Therefore, $\operatorname{Im}(f)$ is a subrepresentation of V_2, and is necessarily V_2 as $f \neq 0$. But if $\operatorname{Ker}(f) = 0$ and $\operatorname{Im}(f) = V_2$, f is a bijective G-linear map from V_1 to V_2, and V_1 and V_2 are isomorphic representations. Contradiction.

(**2**) Suppose that $V_1 = V_2$ and $\rho^1 = \rho^2$. Then, let λ be an eigenvalue of f. Defining $f' := f - \lambda I$, we see that $\rho_s^2 \circ f' = \rho_s^2 \circ f - \lambda \rho_s^2 = f \circ \rho_s^1 - \lambda \rho_s^1 = f' \circ \rho_s^1$. Since λ is an eigenvalue of f, f' has non-zero kernel. $\operatorname{Ker}(f')$ is a nontrivial subrepresentation of V_1, and therefore V_1 itself. Therefore $f' = 0 \Rightarrow f = \lambda I$.

Definition 2.15. Let ρ be a representation. We define the function χ by $\chi(s) = \operatorname{Trace}(\rho_s)$. Then, χ is called the *character* of ρ.

Theorem 2.16 Let χ be the character of a n-dimensional representation V. Then,
(**1**) $\chi(1) = n$
(**2**) $\chi(s^{-1}) = \overline{\chi(s)}$, for all $s \in G$.
(**3**) $\chi(tst^{-1}) = \chi(s)$ for all $s, t \in G$.

Proof. (**1**) $\rho(1)$ is necessarily the identity matrix since $\rho(1)\rho(s) = \rho(s)$ for all $s \in G$. Therefore $\chi(1) =\operatorname{Trace}(I_n) = n$.
(**2**) ρ_s is a matrix of finite order because s is of finite order in G. Therefore, the eigenvalues of ρ_s are also of finite order, so that they are roots of unity. In particular, the eigenvalues of $\rho(s)$, $\lambda_1, ..., \lambda_n$ have absolute value 1. Therefore, for each i, λ_i, $\lambda_i^{-1} = \overline{\lambda_i}$. Therefore, $\operatorname{Trace}(\rho_s^{-1}) = \sum \lambda_i^{-1} = \sum \overline{\lambda_i} = \overline{\operatorname{Trace}(\rho_s)}$.
(**3**) It is known that for any $a, b \in GL(V)$, $\operatorname{Trace}(ab) = \operatorname{Trace}(ba)$. Therefore, setting $a = ts$ and $b = t^{-1}$ implies that $\operatorname{Trace}(tst^{-1}) = \operatorname{Trace}(s)$.

Remark 2.17 A complex-valued function f on G with the property that

$f(s) = f(tst^{-1})$ for all $s, t \in G$ is called a *class function*. By (3) above, any character of a representation is a class function. Moreover, we will show that the set of simple characters of representations of G provide an orthonormal basis for the $\mathbb{C}$-vector space of class functions of G.

Proposition 2.18 Let V and W be representations of G with characters χ_V and χ_W respectively.
(i) $\chi_{V \oplus W} = \chi_V + \chi_W$
(ii) $\chi_{V \otimes W} = \chi_V \cdot \chi_W$

Proof. For any $s \in G$, suppose that the eigenvalues of s in V are $\{\lambda_i\}$ and the eigenvalues of s in W are $\{\mu_j\}$. Then, the eigenvalues of s in $V \oplus W$ are $\{\lambda_i\} \cup \{\mu_j\}$, and the eigenvalues of s in $V \otimes W$ are $\{\lambda_i \cdot \mu_j\}_{i,j}$. Then, $\chi_{V \oplus W} = \sum\limits_i \lambda_i + \sum\limits_j \mu_j = \chi_V + \chi_W$ and $\chi_{V \otimes W} = \sum\limits_i \lambda_i \cdot \sum\limits_j \mu_j = \chi_V \cdot \chi_W$

Definition 2.19 Suppose that ϕ and ψ are complex-valued class functions on G. We define the the inner product $(\cdot|\cdot)$ by

$$(\phi|\psi) := \frac{1}{|G|} \sum_{t \in G} \phi(t)\overline{\psi(t)}$$

Lemma 2.20 For a representation $\rho : G \to GL(V)$, define $V_G := \{v \in V | \rho_g(v) = v, \forall g \in G\}$. Define $\varphi \in \mathrm{End}(V)$ by $\varphi := \frac{1}{|G|} \sum\limits_{g \in G} \rho_g$. Then, $\dim(V_G) = \mathrm{Trace}(\varphi)$.

Proof. The map φ as defined is a projection of V onto V_G. Suppose $v = \frac{1}{|G|} \sum\limits_{g \in G} \rho_g(w)$ for some $w \in V$. Then, for any $h \in G$, $\rho_h(v) = \frac{1}{|G|} \sum\limits_{g \in G} \rho_h \rho_g(w) = \frac{1}{|G|} \sum\limits_{g \in G} \rho_g(w) = v$, so $v \in V_G$. Moreover, if $w \in V_G$, then $\frac{1}{|G|} \sum\limits_{g \in G} \rho_g(w) = w$.
From this information we can view φ as a block matrix of the form $\begin{pmatrix} I_m & A \\ 0 & 0 \end{pmatrix}$ where m is the dimension of V_G, I_m is the identity map representing the projection of V_G to V_G, and A represents the projection of $V \setminus V_G$ to V_G. Therefore $\mathrm{Trace}(\varphi) = m = \dim(V_G)$.

Theorem 2.21 (**Orthogonality Relation**) Let V and W be irreducible

representations of G with characters ϕ and ψ respectively. Let $\mathrm{Hom}_G(V, W)$ be the set of G-linear maps in $\mathrm{Hom}(V, W)$. Then, $(\phi|\psi) = \dim(\mathrm{Hom}_G(V, W))$, the dimension of $\mathrm{Hom}_G(V, W)$ as a $\mathbb{C}$-vector space. From this we get:
(**1**) If χ is the character of an irreducible representation, then $(\chi|\chi) = 1$
(**2**) If χ and χ' are characters of non-isomorphic irreducible representations, then $(\chi|\chi') = 0$

Proof. Let V and W be irreducible representations of G. $\mathrm{Hom}_G(V, W)$ is the set of fixed points of the representation $V^* \otimes W$, which we denote by $(V^* \otimes W)_G$. By the lemma, $\dim((V^* \otimes W)_G) = \mathrm{Trace}(\frac{1}{|G|} \sum\limits_{g \in G} \rho_g^{V^* \otimes W}(w)) = \frac{1}{|G|} \sum\limits_{g \in G} \chi_V(g) \cdot \overline{\chi_W(g)} = (\chi_V|\chi_W)$. Therefore $(\chi_V|\chi_W) = \dim(\mathrm{Hom}_G(V, W))$. By Schur's Lemma, if V and W are isomorphic then $\mathrm{Hom}_G(V, W)$ contains multiples of the identity λI; otherwise, $\mathrm{Hom}_G(V, W)$ contains only the zero map. Therefore, $\dim(\mathrm{Hom}_G(V, W)) = 1$ if $V \simeq W$ and $\dim(\mathrm{Hom}_G(V, W)) = 0$ otherwise.

Corollary 2.22 Let V and W be (not necessarily irreducible) representations of G with characters ϕ and ψ respectively. By the semisimplicity of representations, $(\phi|\psi) = \dim(\mathrm{Hom}_G(V, W))$.

Theorem 2.23 Let W be a representation of G, such that $W = W_1 \oplus ... \oplus W_k$ for some irreducible representations $W_1, ..., W_k$. Then, this direct sum is unique up to isomorphism of the summands.

Proof. Suppose that we can decompose $W = W_1 \oplus W_2 \oplus ... \oplus W_n$ for some irreducible representations $W_1, ..., W_n$. Then the character of W, χ, is equal to $\phi_1 + \phi_2 + ... + \phi_n$ where ϕ_i is the character of W_i. By the previous theorem, the characters of irreducible representations of G are linearly independent, so $\phi_1 + \phi_2 + ... + \phi_n$ is the unique linear combination of irreducible characters equaling χ.

Corollary 2.24 By Theorem, any two representations with the same character are isomorphic.

Theorem 2.25 (**Irreducibility Criterion**). Suppose that V is a representation of G with character χ. $(\chi|\chi) = 1$ iff V is irreducible.

Proof. Suppose V can be decomposed into the direct sum $m_1W_1 \oplus m_2W_2 \oplus ... \oplus m_kW_k$, where the W_i are distinct irreducible representations that appear m_i times in V. We define χ_i to be the character of W_i. Then, the character of V, χ, is equal to $m_1\chi_1 + m_2\chi_2 + ... + m_k\chi_k$ and $(\chi|\chi) = (m_1\chi_1 + m_2\chi_2 + ... + m_k\chi_k | m_1\chi_1 + m_2\chi_2 + ... + m_k\chi_k) = \sum_i m_i^2$. $\sum_i m_i^2 = 1$ iff exactly one of the m_i is 1 and the rest are 0.

Theorem 2.26 The set of characters of irreducible representations of G form an orthonormal basis of the vector space of complex-valued class functions of G.

Proof. We have seen that the set of characters of irreducible representations of G forms an orthonormal set of class functions. So it suffices to prove that there is no class function outside the span of the irreducible characters of representations of G. Let α be a class function of G and $(\alpha|\chi_V) = 0$ for all characters χ_V of irreducible representations V. We will show that $\alpha = 0$. Let $\rho : G \to GL(V)$ be a representation of G and define $\varphi_V \in \mathrm{End}(V)$ by $\varphi_V := \sum_{g\in G} \alpha(g)\rho_g$. Then, φ_V is a G-linear map:

$$\varphi_V(\rho_h(v)) = \sum_{g\in G} \alpha(g)\rho_{gh}(v) = \sum_{g\in G} \alpha(hgh^{-1})\rho_{hg}(v)$$
$$= \rho_h(\sum_{g\in G} \alpha(hgh^{-1})\rho_g(v)) = \rho_h(\varphi_V(v))$$

Now suppose that V is an irreducible representation of G. By Schur's Lemma, $\varphi_V \lambda \cdot Id$ for some $\lambda \in \mathbb{C}$. If $\dim(V) = n$, then we find that $\lambda = \frac{1}{n}\,\mathrm{Trace}(\varphi_V) = \frac{1}{n}\sum_{g\in G} \alpha(g)\chi_V(g) = \frac{1}{n}\sum_{g\in G} \alpha(g)\overline{\chi_{V^*}(g)} = \frac{|G|}{n}(\alpha|\chi_{V^*})$ by the definition of the dual representation. But since α is outside of the span of the irreducible characters of representations of G, $\lambda = \frac{|G|}{n}(\alpha|\chi_{V^*}) = 0$. Since $\varphi_V = 0$ for all irreducible representations V, $\varphi_V = 0$ for all representations of G by the semisimplicity of representations. We apply this to the regular representation. Recall that the regular representation has a basis $\{e_t\}_{t\in G}$ satisfying $\rho_s(e_t) = e_{st}$. Then, $\sum_{g\in G} \alpha(g)\rho_g(e_1) = \sum_{g\in G} \alpha(g)e_g = 0$. By the linear independence of the basis $\{e_t\}_{t\in G}$, $\alpha(g) = 0$ for all $g \in G$.

Corollary 2.27 In particular, the number of irreducible representations of G is the number of conjugacy classes of G.

Proof. The set of class functions $\{f_c\}$ that take the value 1 on a conjugacy class c and 0 elsewhere form a basis of the vector space of complex-valued class functions of G. Therefore, the space of class functions of G has dimension equal to the number of conjugacy classes of G.

Definition 2.28 Let $\rho : G \to GL(V)$ be a representation of G in V. Suppose that H is a subgroup of G. Then, the *restriction* of ρ to H is the representation $\rho_H : H \to GL(V)$ defined by $\rho_s^H := \rho_s$ for all $s \in H$.

Definition 2.29 Let $\rho : G \to GL(V)$ be a representation of G in V. Suppose that H is a subgroup of G, and ρ^H is the restriction of ρ to H. Let W be a subrepresentation of ρ^H. Then for every left coset of H, $\sigma = sH$, $W_\sigma := \rho_s W$ is a subrepresentation of ρ. Therefore, $\sum\limits_{\sigma \in G/H} W_\sigma$ is a subrepresentation of V. A representation ρ of G in V is said to be *induced* by a representation θ of G in W if $V = \oplus_{\sigma \in G/H} W_\sigma$.

Remark 2.30 Let H be a subgroup of G. Suppose we have a representation $\theta : H \to GL(W)$. Then there exists a unique representation of G that is induced by θ.

Theorem 2.31 Let $\rho : G \to GL(V)$ be the representation induced by $\theta : H \to GL(W)$. Then, for any $t \in G$, $\chi_\rho(t) = \frac{1}{|H|} \sum\limits_{s \in G | sts^{-1} \in H} \chi_\theta(sts^{-1})$.

Proof. Since $\rho_g W_\sigma = W_{g\sigma}$ for all $g \in G$, we can calculate the trace of ρ_g by taking the trace of θ_g over the cosets σ such that $g\sigma = \sigma$, since the matrix representation of θ_g on those cosets can be seen on the diagonal of the matrix representation of ρ_g. If σ is the coset sH, then $g\sigma = \sigma$ is equivalent to $sgs^{-1} \in H$. Therefore, $\chi_\rho(t) = \sum\limits_{sH \in G/H | sts^{-1} \in H} \chi_\theta(sts^{-1}) = \frac{1}{|H|} \sum\limits_{s \in G | sts^{-1} \in H} \chi_\theta(sts^{-1})$.

Theorem 2.32 (**Frobenius Reciprocity**) Suppose that ϕ is a character of a representation V of G and ψ is a character of a representation W of H. We will denote by $\mathrm{res}\phi$ the character of the restriction to H, and by $\mathrm{ind}\psi$ the character of the induced representation to G. Then,

$$(\psi|\text{res}\phi)_H = (\text{ind}\psi|\phi)_G$$

.

Proof. We prove that $\dim(\text{Hom}_H(W, \text{res}V)) = \dim(\text{Hom}_G(\text{ind}W, V))$. Since $\text{ind}W = \sum_{\sigma \in G/H} W_\sigma$, every $\varphi \in \text{Hom}_H(W, \text{res}V)$ extends uniquely to a homomorphism $\varphi' \in \text{Hom}_G(\text{ind}W, V)$ by the relation $\varphi' = \rho_g^V \circ \varphi \circ (\rho_g^{\text{ind}W})^{-1}$ on each of the direct summands W_{gH}. Therefore, $\text{Hom}_H(W, \text{res}V) = \text{Hom}_G(\text{ind}W, V)$ and $\dim(\text{Hom}_H(W, \text{res}V)) = \dim(\text{Hom}_G(\text{ind}W, V))$.

3 Cartan Subgroups

Let F be a field. Suppose that K is a separable quadratic extension of F. Let $\{\omega_1, \omega_2\}$ be some basis of K/F. The regular representation of K with respect to the basis is multiplication representing $K^\times$ as a subgroup of $GL_2(F)$.

Definition 3.1 Let C_k denote the image of $K^\times$ in $GL_2(F)$ under the regular representation of K with respect to some K/F basis. C_k is called a *non-split Cartan subgroup.*

Remark 3.2 A change the basis for the regular representation of K corresponds to conjugation of the subgroup C_k, so all non-split Cartan subgroups are conjugate to each other.

Remark 3.3 The subalgebra $F[C_k] \subseteq \mathrm{Mat}_2(F)$ is isomorphic to K.

Example 3.4 Let $F = \mathbb{R}$ and $K = \mathbb{C}$. Define $\rho : \mathbb{C}^\times \to GL_2(\mathbb{R})$ to be the regular representation of $\mathbb{C}$ with respect to the basis $\{1, i\}$. In general $\rho(a + bi) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, where $a, b \in \mathbb{R}$.

Definition 3.5 The *split Cartan subgroup* is the group A of diagonal matrices

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

with $a, d \in F^\times$

Proposition 3.6 Let C_k be a non-split Cartan subgroup of $GL_2(F)$. The subgroup C_k is maximal among commutative subgroups of $GL_2(F)$.

Proof. Since C_k is isomorphic to $K^\times$, C_k is commutative. Suppose that there is an $\alpha \in GL_2(F) \setminus C_k$ that commutes with all elements in C_k. In this case, the set $\{1, \alpha\}$ is linearly independent over $F[C_k]$. $F[C_k] \cong K$ may be viewed as an F-vector space of dimension 2, so $F[C_k] + \alpha F[C_k]$ can be viewed as a 4-dimensional F-vector space, from which it follows that it must be the entire set $\mathrm{Mat}_2(F)$. This implies that $\mathrm{Mat}_2(F)$ is commutative, which is false.

Proposition 3.7 Let A be the split Cartan subgroup of $GL_2(F)$. The subgroup A is maximal among commutative subgroups of $GL_2(F)$.

Proof. Since the split Cartan subgroup is the subgroup of diagonal matrices of $GL_2(F)$, it is a commutative subgroup. Let $\alpha \in GL_2(F) \setminus A$. We write $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying the condition that either b or c is non-zero. Then, for some diagonal matrix $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ with $x \neq y$ in the split Cartan subgroup, we have that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} ax & by \\ cx & dy \end{pmatrix}$$

while

$$\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ax & bx \\ cy & dy \end{pmatrix}$$

Since $x \neq y$ and either b or c is non-zero, either $bx \neq by$ or $cx \neq cy$, proving that α does not commute with the rest of A.

Proposition 3.8 Let C be a Cartan subgroup of $GL_2(F)$. Then, $[N(C) : C] = 2$, where $N(C)$ is the normalizer of C

Proof. Conjugation by a member of $N(C_k)$ on a non-split Cartan subgroup C_k can be viewed as an automorphism of K/F. Since C_k is maximal commutative, conjugation by elements of C_k can be viewed as the trivial automorphism. Therefore, there must be a coset of C_k in $N(C_k)$ that can be viewed as the unique non-trivial automorphism of K/F. We see that $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ generates this coset by interchanging the eigenspaces of the elements of C_k. Similarly, w interchanges the eigenpaces of the split Cartan A, sending a matrix $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mapsto \begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}$. This must generate a unique non-trivial coset of A in $N(A)$ because matrices of the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ can only be conjugate to themselves or to matrices of the form $\begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}$ in A.

We now specialize to the scenario of $GL_2(F)$ where F is a finite field, which is the topic of this thesis. Going forward, we use the following notation:

$F =$ finite field with q elements, where q is a power of some prime p
$G = GL_2(F)$
$Z = Z(G)$, the center of G
$A =$ diagonal subgroup of G (split Cartan)
$C =$ a non-split Cartan subgroup of G. **Note**. Since F is a finite field, there is a unique quadratic extension of F in its algebraic closure; up to conjugacy there is one non-split Cartan subgroup.
$U =$ the group of unipotent elements $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$
$B =$ Borel subgroup $UA = AU$. The Borel subgroup consists of all matrices of the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with $a, c \in F^\times$ and $b \in F$.

Remark 3.9 From the definitions, we have that the size of $G = GL_2(F)$ is $(q^2 - q)(q^2 - 1)$. The size of the diagonal subgroup A is $(q-1)^2$. The size of a non-split Cartan subgroup C is $q^2 - 1$. The size of the Borel subgroup B is $q(q-1)^2$.

4 Conjugacy Classes of $\mathbf{GL}_2(\mathbb{F}_q)$

We determine the conjugacy classes of G.

Theorem 4.1. The conjugacy classes of G are given by the following table.

Class type	# of classes	# of elements of class
$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$	$q-1$	1
$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$	$q-1$	q^2-1
$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ with a $\neq$ d	$\frac{1}{2}(q-1)(q-2)$	q^2+q
$\alpha \in C - Z$	$\frac{1}{2}(q-1)q$	q^2-q

Table 1: Conjugacy Classes of G

Proof. Suppose $\alpha \in G$ has characteristic polynomial with roots in F; in other words, the eigenvalues of α are in F. Then, by the Jordan canonical form, such a value of α is conjugate to one of the matrices $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$, or $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ with $a \neq d$.

On the other hand, elements $\alpha \in C - Z$ correspond to members of $K \setminus F$, and these elements do not have a characteristic polynomial that splits over F.

To complete the table we have two objectives:

(1) Find the number of classes in each type. We will observe which elements in a type are conjugate to each other by using the characteristic polynomial, which is unchanged by conjugation.
(2) Find the number of elements in a conjugacy class for each type. Since each of $\beta\alpha\beta^{-1}$ are conjugate to α for $\alpha, \beta \in G$, the number of elements in a conjugacy class can be counted as the index of the centralizer of the element in $GL_2(F)$.

First class type $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

(1) The characteristic polynomial is $(x-a)^2$, so each a determines a distinct

conjugacy class. Therefore there are $q-1$ conjugacy classes of the first type.
(2) The first type of class consists of the central elements of G. Since the centralizer of an element in a conjugacy class of the first type is the entire group G. Therefore the number of elements in a class of this type is $[G:G]=1$.

Second class type $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$.

(1) The characteristic polynomial is $(x-a)^2$, so each a determines a distinct conjugacy class. Therefore there are $q-1$ conjugacy classes of the second type.
(2) We explicitly calculate the centralizer of a given matrix of the second type. $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$ is in the centralizer of matrix $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ iff

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

which implies that

$$\begin{pmatrix} a\alpha & \alpha + a\beta \\ a\gamma & \gamma + a\delta \end{pmatrix} = \begin{pmatrix} a\alpha + \gamma & \delta + a\beta \\ a\gamma & a\delta \end{pmatrix}$$

Therefore, $\gamma = 0$ and $\delta = \alpha$. Since $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$ is invertible, its determinant α^2 is non-zero. Therefore the centralizer of matrix $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ is the set of matrices $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$ such that $\alpha \neq 0$. There are $q(q-1)$ such matrices, so the index of this centralizer in G is $\frac{q(q-1)(q^2-1)}{q(q-1)} = q^2 - 1$.

Third class type $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ with $a \neq d$.

(1) The characteristic polynomial of a given matrix of this type is $(x-a)(x-d)$. We see that $\begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}$ is the only other matrix of this type with the same characteristic polynomial. Furthermore, $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Therefore, every polynomial $(x-a)(x-d)$ with $a, d \in F^{\times}$and $a \neq d$ is the characteristic polynomial of two conjugate matrices of the third type. Hence, the number of distinct classes in the third class type is $\frac{(q-1)(q-2)}{2}$.

(2) As in the case of the second type, we directly calculate the centralizer of this type of conjugacy class. $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$ is in the centralizer of matrix $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ iff

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

which implies that

$$\begin{pmatrix} a\alpha & d\beta \\ a\gamma & d\delta \end{pmatrix} = \begin{pmatrix} a\alpha & a\beta \\ d\gamma & d\delta \end{pmatrix}$$

Therefore, $d\gamma = a\gamma$ and $d\beta = a\beta$. Since $a \neq d$, this implies that $\gamma = \beta = 0$. Since $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$ is invertible, its determinant $\alpha\delta$ is non-zero. Therefore the centralizer of matrix $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ is the set of matrices $\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$ such that $\alpha \neq 0$ and $\delta \neq 0$. There are $(q-1)^2$ such matrices, so the index of this centralizer in G is $\frac{q(q-1)(q^2-1)}{(q-1)^2} = q^2 + q$.

Fourth class type $\alpha \in C - Z$.

(1) $|C - Z| = q^2 - q$. Conjugation of C corresponds to a field automorphism of K, which is either the identity map or Frobenius map. For example, conjugation of C by the identity matrix corresponds to the identity map on K, while conjugation by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ corresponds to the Frobenius map. So two distinct matrices in $C - Z$ are conjugate iff their preimages in K are related by the Frobenius map. Therefore, there are $\frac{q^2-q}{2}$ classes of the fourth type.

(2) Since a non-split Cartan subgroup C is commutative, the centralizer of any element in $C - Z$ is C itself. Since $|C| = q^2 - 1$, the number of elements in the class of some $\alpha \in C - Z$ is $[G : C] = \frac{q(q-1)(q^2-1)}{q^2-1} = q^2 - q$.

Finally, we show that our findings give all the conjugacy classes of G. The total number of elements of G that are conjugate to one of the classes of a certain class type is the product of the number of classes and number of elements of the class. Therefore, the total number of elements of G that are conjugate to a class in one of the four described class types is $(q-1)(1) + (q-1)(q^2-1) + (\frac{(q-1)(q-2)}{2})(q^2+q) + (\frac{(q-1)q}{2})(q^2-q) = q(q-1)(q^2-1) = |G|$.

5 1-Dimensional Representations

Theorem 5.1 Let $\mu : F^\times \to \mathbb{C}^\times$ be a group homomorphism. Then, functions on G of the type $\mu \circ \det : G \to \mathbb{C}^\times$ are characters of irreducible 1-dimensional representations.

Proof. $\mu \circ \det$ is is a homomorphism from G to $GL(\mathbb{C}) \simeq \mathbb{C}^\times$, so it is a 1-dimensional representation. It is necessarily irreducible since it is 1-dimensional.

The character of a 1-dimensional representation takes the following values on the conjugacy classes of G:

χ/Conjugacy Class	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$	$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ $d \neq a$	$\alpha \in C - Z$
$\mu \circ det$	$\mu(a)^2$	$\mu(a)^2$	$\mu(ad)$	$\mu \circ det(\alpha)$

Table 2: Character of a 1-Dimensional representation

Theorem 5.2 There are $q-1$ irreducible 1-dimensional representations.

Proof. There are $q-1$ homomorphisms of the type μ, that are determined by the $q-1$th root of unity that a generator of $F^\times$ is mapped to.

6 Steinberg Representations

Since $U \lhd B$, $B/U \simeq A$ and therefore a character of A can be viewed as a character of B by the quotient map. Specifically, we let $\psi_\mu = \mathrm{res_A}(\mu \circ \det)$. We view ψ_μ as a character on B, so that for example $\psi_\mu \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \mu(ad)$.
From this character we obtain an induced character $\psi_\mu^G = \mathrm{ind}_B^G(\psi_\mu)$.
This character contains $\mu \circ \det$ and is therefore not irreducible, which we see by Frobenius reciprocity:

$$(\mathrm{ind}_B^G \psi_\mu | \mu \circ \det)_G = (\psi_\mu | \mu \circ \det)_B = \frac{1}{|B|} \sum_{t \in B} |\mu \circ \det(t)|^2 = 1$$

.

Theorem 6.1 Characters $\chi = \psi_\mu^G - \mu \circ \det$ are characters of irreducible representations. Specifically, we call the representation that they characterize the Steinberg representations.

Proof. In order to find the values taken by ψ_μ^G in each conjugacy class, we use the formula $\mathrm{ind}_B^G(\psi_\mu)(\alpha) = \frac{1}{|B|} \sum\limits_{t \in G | t\alpha t^{-1} \in B} \psi_\mu(t\alpha t^{-1})$.

First class type $\alpha = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

The t such that $t\alpha t^{-1} \in B$ is the entire group G since the first class type consists of the central elements of G. $\frac{1}{|B|} \sum\limits_{t \in G | t\alpha t^{-1} \in B} \psi_\mu(t\alpha t^{-1}) = \frac{1}{|B|} \sum\limits_{t \in G} \psi_\mu(\alpha) = \frac{|G|}{|B|}\mu(a)^2 = \frac{(q^2-1)(q^2-q)}{(q-1)(q^2-q)}\mu(a)^2 = (q+1)\mu(a)^2$. Furthermore, $\mu \circ \det(\alpha) = \mu(a)^2$, so $(\psi_\mu^G - \mu \circ \det)(\alpha) = q\mu(a)^2$

Second class type $u = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$.

First we find the $t \in G$ such that $tut^{-1} \in B$. $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$ is such a t iff

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

for some $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in B$. This implies

$$\begin{pmatrix} a\alpha & \alpha + a\beta \\ a\gamma & \gamma + a\delta \end{pmatrix} = \begin{pmatrix} x\alpha + y\gamma & x\beta + y\delta \\ z\gamma & z\delta \end{pmatrix}$$

Therefore, $z\gamma = a\gamma$, implying either that $\gamma = 0$ or that $a = z$. But if $a = z$, then $\gamma + a\delta = z\delta$ which implies that γ is 0 anyways. Therefore, the set of matrices t for which $tut^{-1} \in B$ is the set of matrices $\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$ with $\alpha, \delta \in F^\times$ since t is invertible. We then notice that the t satisfying these requirements is precisely the Borel subgroup. Therefore, $\frac{1}{|B|} \sum\limits_{t \in G | tut^{-1} \in B} \psi_\mu(tut^{-1}) = \frac{|B|}{|B|}\mu(a)^2$. Furthermore, $\mu \circ \det(u) = \mu(a)^2$, so $(\psi_\mu^G - \mu \circ \det)(u) = 0$

Third class type $s = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ with $a \neq d$.

First we find the t such that $tst^{-1} \in B$. $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$ is such a t iff

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

for some $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in B$. This implies

$$\begin{pmatrix} a\alpha & d\beta \\ a\gamma & d\delta \end{pmatrix} = \begin{pmatrix} x\alpha + y\gamma & x\beta + y\delta \\ z\gamma & z\delta \end{pmatrix}$$

γ and δ are not both 0 since t is invertible. Therefore, either $z = d$ or $z = a$. If $z = d$, then $\gamma = 0$. On the other hand, if $z = a$, then $\delta = 0$. Therefore, the matrices t are of either the form $\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$ with $\alpha, \delta \in F^\times$ or of the form $\begin{pmatrix} \alpha & \beta \\ \gamma & 0 \end{pmatrix}$ with $\beta, \gamma \in F^\times$. There are $2q(q-1)^2$ such matrices so we get that $\frac{1}{|B|} \sum_{t \in G | tst^{-1} \in B} \psi_\mu(tst^{-1}) = \frac{2q(q-1)^2}{|B|}\mu(ad) = 2\mu(ad)$. Furthermore, $\mu \circ \det(s) = \mu(ad)$, so $(\psi_\mu^G - \mu \circ \det)(u) = \mu(ad)$.

Fourth class type $\alpha \in C - Z$.

There is no $t \in G$ such that $t\alpha t^{-1}$ is in B; we can see this from the fact that matrices in $C - Z$ don't have eigenvalues inside F while matrices in B have eigenvalues in F. Therefore, $(\psi_\mu^G - \mu \circ \det)(\alpha) = 0 - \mu \circ \det(\alpha)$.

Therefore the character of a Steinberg representation takes the following values on the conjugacy classes of G:

χ/Conjugacy Class	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$	$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ $d \neq a$	$\alpha \in C - Z$
$\psi_\mu^G - \mu \circ det$	$q\mu(a)^2$	0	$\mu(ad)$	$-\mu \circ det(\alpha)$

Table 3: Character of a Steinberg representation

We calculate $((\psi_\mu^G - \mu\circ\det)|(\psi_\mu^G - \mu\circ\det))$, and use the irreducibility criterion to determine that we have found a character of an irreducible representation.

$((\psi_\mu^G - \mu \circ \det)|(\psi_\mu^G - \mu \circ \det)) = \frac{1}{|G|}\sum_{t\in G}|(\psi_\mu^G - \mu \circ \det)(t)|^2$; we can use Table 3 to find this sum over the elements of each class type.

$\sum_{t\in\text{First class type}} |(\psi_\mu^G - \mu \circ \det)(t)|^2 = \sum_{a\in F^\times} |q\mu(a)|^2 = (q-1)(q^2)$.
$\sum_{t\in\text{Second class type}} |(\psi_\mu^G - \mu \circ \det)(t)|^2 = \sum_{a\in F^\times} (q^2-1)|0|^2 = 0$.
$\sum_{t\in\text{Third class type}} |(\psi_\mu^G - \mu\circ\det)(t)|^2 = \frac{1}{2}\sum_{\{a,d\}\in F^\times|a\neq d}(q^2+q)|\mu(ad)|^2 = \frac{(q-1)(q-2)}{2}(q^2+q)$.
$\sum_{t\in\text{Fourth class type}} |(\psi_\mu^G - \mu \circ \det)(t)|^2 = \frac{1}{2}\sum_{\alpha\in C-Z}(q^2-q)|-\mu\circ\det(\alpha)|^2 = \frac{1}{2}(q^2-q)(q^2-q)$

The value $((\psi_\mu^G - \mu \circ \det)|(\psi_\mu^G - \mu \circ \det)) = \frac{1}{|G|}[(q-1)q^2 + \frac{1}{2}(q-1)(q-2)(q^2+q) + \frac{1}{2}(q-1)q(q^2-q)] = 1$.

Theorem 6.2 There are $q-1$ irreducible Steinberg representations. They are of dimension q.

Proof. Just as with the 1-dimensional representations, the Steinberg representations are determined by the homomorphism μ. There are $q-1$ homomorphisms of the type μ, which are determined by the $q-1$th root of unity that a generator of $F^\times$ is mapped to.
The dimension of ψ_μ^G is $\frac{|G|}{|B|}(1) = q+1$ by the direct sum decomposition of an induced representation. Since we remove a 1-dimensional character $\mu\circ\det$, the dimension of a Steinberg representation is q.

7 Principal Series Representations

Let $\psi : A \to \mathbb{C}^\times$ be a homomorphism.
Let $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then we have that $w \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} w^{-1} = \begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}$.
We define the conjugate character $[w]\psi := \psi(waw^{-1})$. Recall that the characters $\mu \circ \det$ are homomorphisms from A to $\mathbb{C}^\times$. Then, $[w](\mu \circ \det) = \mu \circ \det$ since the determinant is unchanged under conjugation. On the other hand, if ψ is a homomorphism from A to $\mathbb{C}^\times$ and $\psi \neq \mu \circ \det$, then ψ is necessarily distinct from $[w]\psi$. In this case, ψ takes the form such that $\psi \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \psi_1(a)\psi_2(d)$ for two distinct homomorphisms ψ_1 and ψ_2 from $F^\times \to \mathbb{C}^\times$.

We view ψ as a character on B, so that for example $\psi \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \psi_1(a)\psi_2(d)$.
We can now induce a character from B to G, defining $\psi^G = \operatorname{ind}_B^G(\psi) = \operatorname{ind}_B^G([w]\psi)$, with ψ satisfying $\psi \neq [w]\psi$.

Theorem 7.1. Characters $\chi = \psi^G$ where $\psi \neq [w]\psi$ are characters of irreducible representations. We call the representations that they characterize the Principal Series representations.

Proof. In order to find the values taken by ψ^G in each conjugacy class, we use the formula $\operatorname{ind}_B^G(\psi)(\alpha) = \frac{1}{|B|} \sum_{t \in G | t\alpha t^{-1} \in B} \psi(\beta\alpha\beta^{-1})$.

First class type $\alpha = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

The t such that $t\alpha t^{-1} \in B$ is the entire group G. $\psi^G(\alpha) = \frac{1}{|B|} \sum_{t \in G | t\alpha t^{-1} \in B} \psi(t\alpha t^{-1}) = \frac{1}{|B|} \sum_{t \in G | t\alpha t^{-1} \in B} \psi(\alpha) = \frac{|G|}{|B|}\psi_1\psi_2(a) = (q+1)\psi_1\psi_2(a)$.

Second class type $u = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$.

Recall from the previous section that $tut^{-1} \in B$ iff $t \in B$. Therefore, $\psi^G(u) = \frac{1}{|B|} \sum_{t \in G | t\alpha t^{-1} \in B} \psi(t\alpha t^{-1}) = \frac{1}{|B|} \sum_{t \in B} \psi(\alpha) = \psi_1\psi_2(a)$.

Second class type $s = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ with $a \neq d$.

Recall that the t such that $t\alpha t^{-1} \in B$ are matrices of the form $\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \in G$ if the conjugate matrix is of the form $\begin{pmatrix} a & y \\ 0 & d \end{pmatrix}$. There are $q(q-1)^2$ such matrices. Alternatively, t is of the form $\begin{pmatrix} \alpha & \beta \\ \gamma & 0 \end{pmatrix} \in G$ if the conjugate matrix is of the form $\begin{pmatrix} d & y \\ 0 & a \end{pmatrix}$. There are $q(q-1)^2$ such matrices.
Therefore, $\psi^G(s) = \frac{1}{|B|}q(q-1)^2\psi_1(a)\psi_2(d) + \frac{1}{|B|}q(q-1)^2\psi_1(d)\psi_2(a) = \psi_1(a)\psi_2(d) + \psi_1(d)\psi_2(a)$.

Fourth class type $\alpha \in C - Z$.

Recall that there are no $t \in G$ such that $t\alpha t^{-1} \in B$. Therefore, $\psi^G(\alpha) = 0$.

Therefore the character of a Principal Series representation takes the following values on the conjugacy classes of G:

χ/Conjugacy Class	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$	$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ $a \neq d$	$\alpha \in C - Z$
ψ^G with $\psi \neq [w]\psi$	$(q+1)\psi_1\psi_2(a)$	$\psi_1\psi_2(a)$	$\psi_1(a)\psi_2(d) + \psi_1(d)\psi_2(a)$	0

Table 4: Character of a Principal Series representation

We calculate $(\psi^G|\psi^G)$, and use the irreducibility criterion to determine that we have found a character of an irreducible representation.

$(\psi^G|\psi^G) = \frac{1}{|G|}\sum\limits_{t\in G}|\psi^G(t)|^2$; we can use Table 4 to find this sum over the elements of each class type.

$\sum\limits_{t\in\text{First class type}}|\psi^G(t)|^2 = \sum\limits_{a\in F^\times}|(q+1)\psi_1\psi_2(a)|^2 = (q-1)(q+1)^2$.
$\sum\limits_{t\in\text{Second class type}}|\psi^G(t)|^2 = \sum\limits_{a\in F^\times}(q^2-1)|\psi_2\psi_2(a)|^2 = (q-1)(q^2-1)$.
$\sum\limits_{t\in\text{Third class type}}|\psi^G(t)|^2 = \frac{1}{2}\sum\limits_{a,d\in F^\times|a\neq d}(q^2+q)|\psi_1(a)\psi_2(d)+\psi_1(d)\psi_2(a)|^2 = (q^2+1)(q-1)(q-3)$.

$$\sum_{t \in \text{Fourth class type}} |\psi^G(t)|^2 = \frac{1}{2} \sum_{\alpha \in C - Z} (q^2 - q)|0|^2 = 0$$

The value $(\psi^G|\psi^G) = \frac{1}{|G|}[(q-1)(q+1)^2 + (q-1)(q^2-1) + (q^2+1)(q-1)(q-3)] = 1$.

Theorem 7.2 There are $\frac{1}{2}(q-1)(q-2)$ irreducible Principal Series representations. They are of dimension $q+1$.

Proof. The Principal Series representations are determined by two distinct homomorphisms ψ_1 and ψ_2 that map $F^\times$ to $\mathbb{C}^\times$. This gives $q-1$ homomorphisms ψ_1 multiplied by $q-2$ remaining options for ψ_2. However, we can observe from the character table that switching the homomorphisms gives the same character, so there are $\frac{1}{2}(q-1)(q-2)$ Principal Series representations.
The dimension of ψ^G is $\frac{|G|}{|B|}(1) = q+1$ by the direct sum decomposition of an induced representation.

8 Cuspidal Representations

Let $\theta : K^\times \to \mathbb{C}^\times$ be a homomorphism. We may view this as a character on a non-split Cartan subgroup C.
Let $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $w \in N(C)$.
The map $\phi : C \to C$ mapping $\alpha \mapsto w\alpha w$ is therefore an automorphism of C. Moreover, ϕ defines a field automorphism of $F[C] \cong K$ over F, which is given by the Frobenius map $\alpha \mapsto \alpha^q$. We define a conjugate character $[w]\theta$ by $[w]\theta = \theta(\phi(\alpha))$. From θ we induce a character $\theta^G = \mathrm{ind}_C^G(\theta) = \mathrm{ind}_C^G([w]\theta)$.

Let $\mu : F^\times \to \mathbb{C}^\times$ be a homomorphism, and let $\lambda : F^+ \to \mathbb{C}^\times$ be a non-trivial homomorphism.

We define (μ, λ) to be the character on ZU such that $(\mu, \lambda)(\begin{pmatrix} a & ax \\ 0 & a \end{pmatrix}) = \mu(a)\lambda(x)$. Then, we induce a character $(\mu, \lambda)^G = \mathrm{ind}_{ZU}^G(\mu, \lambda)$.

Next we calculate the character table for θ^G and $(\mu, \lambda)^G$ as an intermediate step; these are not necessarily characters of irreducible representations.

Theorem 8.1. The value of the characters θ^G and $(\mu, \lambda)^G$ on conjugacy classes of G is given in the following table.

χ/Conjugacy Class	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$	$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ $a \neq d$	$\alpha \in C - Z$
θ^G	$(q^2 - q)\theta(a)$	0	0	$\theta(\alpha) + \theta(w\alpha w)$
$(\mu, \lambda)^G$	$(q^2 - 1)\mu(a)$	$-\mu(a)$	0	0

Table 5: Intermediate Characters

Proof. In order to find the values taken by θ^G in each conjugacy class, we use the formula $\mathrm{ind}_C^G(\theta)(\alpha) = \frac{1}{|C|} \sum\limits_{t \in G | t\alpha t^{-1} \in C} \theta(t\alpha t^{-1})$. In order to find the values taken by $(\mu, \lambda)^G$ in each conjugacy class, we use the formula $\mathrm{ind}_{ZU}^G((\mu, \lambda))(\alpha) = \frac{1}{|ZU|} \sum\limits_{t \in G | t\alpha t^{-1} \in ZU} \theta(t\alpha t^{-1})$.

First class type $\alpha = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

$\theta^G(\alpha) = \frac{1}{|C|} \sum_{t \in G | t\alpha t^{-1} \in C} \theta(\alpha) = \frac{|G|}{|C|}\theta(a) = \frac{(q^2-q)(q^2-1)}{(q^2-1)} = (q^2-q)\theta(a)$.

$(\mu, \lambda)^G = \frac{1}{|ZU|} \sum_{t \in G | t\alpha t^{-1} \in ZU} (\mu, \lambda)(\alpha) = \frac{|G|}{|ZU|}(\mu, \lambda)(\alpha) = \frac{|G|}{|ZU|}\mu(a)\lambda(0) = \frac{|G|}{|ZU|}\mu(a) = \frac{(q^2-q)(q^2-1)}{(q-1)q}\mu(a) = (q^2-1)\mu(a)$.

Second class type $u = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$.

Elements of the second class type are not conjugate to any element of C, so $\theta^G(u) = 0$.

We want to find $t \in G$ such that $t \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} t^{-1} \in ZU$. We notice that the characteristic polynomial of α is $(\lambda - a)^2$, so α should only be conjugate to elements in ZU of the form $\begin{pmatrix} a & ax \\ 0 & a \end{pmatrix}$ where $x \neq 0$. In order for this to occur, we need to find $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$ such that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & ax \\ 0 & a \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

this implies

$$\begin{pmatrix} a\alpha & \alpha + a\beta \\ a\gamma & \gamma + a\delta \end{pmatrix} \begin{pmatrix} a\alpha + ax\gamma & a\beta + ax\delta \\ a\gamma & a\delta \end{pmatrix}$$

Then, we must have that $\gamma = 0$ and $ax\delta = \alpha$. This gives $(q-1)q$ values for β for each $x \in F^\times$. $\frac{1}{|ZU|} \sum_{t \in G | t\alpha t^{-1} \in ZU} (\mu, \lambda)(\alpha) = \frac{q(q-1)}{|ZU|} \sum_{x \in F^+ \setminus \{0\}} (\mu, \lambda)(\alpha) = \frac{q(q-1)}{|ZU|}\mu(a) \sum_{b \in F^\times} \lambda(b) = -\mu(a)$. This follows from the face that $\sum_{b \in F^\times} \lambda(b) = -1$. Since λ is non-trivial, there is a $\lambda(a) \neq 1$. Then, $\lambda(a) \sum_{b \in F^+} \lambda(b) = \sum_{b \in F^+} \lambda(a + b) = \sum_{b \in F^+} \lambda(b)$, which implies that $(\lambda(a) - 1) \sum_{b \in F^+} \lambda(b) = 0$ so $\sum_{b \in F^+} \lambda(b) = 0$. $\sum_{b \in F^\times} \lambda(b) = \sum_{b \in F^+} \lambda(b) - \lambda(0) = -1$.

Third class type $s = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ with $a \neq d$.

Again, elements of the third class type are not conjugate to any element of C, so $\theta^G(s) = 0$.

Since the characteristic polynomial of elements of ZU are of the form $(x-a)^2$, elements of the third class type of G cannot be conjugate to any element of ZU. Therefore, $(\mu, \lambda)^G(s) = 0$

Fourth class type $\alpha \in C - Z$.

Since C is of index two in its normalizer, with the matrix $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ generating the non-trivial coset of C in $N(C)$, $\frac{1}{|C|}\sum_{t\in C}\theta(t\alpha t^{-1}) + \frac{1}{|C|}\sum_{t\in wC}\theta(t\alpha t^{-1}) = \theta(\alpha) + \theta(w\alpha w)$.

Since the characteristic polynomial of elements of the fourth class type is not $(x-a)^2$ for some $a \in F^{\times}$, elements of this type are never conjugate to elements in ZU. Therefore, $(\mu, \lambda)^G(\alpha) = 0$.

Theorem 8.2
Let $\theta\prime$ be a character of the form $(res\theta, \lambda)^G - \theta^G$ where $\theta\prime \neq [w]\theta$. Then $\theta\prime$ is a character of an irreducible representation. We call the representation it characterizes a Cuspidal representation.
Proof. Finding the character table for $\theta\prime$ is easy since we have the partial results from the previous table; we need to subtract the entries of the first row from the second, evaluated at $\mu = \text{res}\theta$.

χ/Conjugacy Class	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$	$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ $d \neq a$	$\alpha \in C - Z$
$(res\theta, \lambda)^G - \theta^G$	$(q-1)\theta(a)$	$-\mu(a)$	0	$-\theta(\alpha) - \theta(w\alpha w)$

Table 6: Character of a Cuspidal representation

We calculate $(\theta'|\theta')$, and use the irreducibility criterion to determine that we have found a character of an irreducible representation.

$(\theta'|\theta') = \frac{1}{|G|}\sum\limits_{t\in G}|\theta'(t)|^2$; we can use Table 6 to find this sum over the elements of each class type.

$$\sum_{t\in\text{First class type}} |\theta'(t)|^2 = \sum_{a\in F^\times} |(q-1)\theta(a)|^2 = (q-1)(q-1)^2.$$
$$\sum_{t\in\text{Second class type}} |\theta'(t)|^2 = \sum_{a\in F^\times} (q^2-1)|-\theta(a)|^2 = (q-1)(q^2-1).$$
$$\sum_{t\in\text{Third class type}} |\theta'(t)|^2 = \frac{1}{2}\sum_{a,d\in F^\times|a\neq d} (q^2+q)|0|^2 = 0.$$
$$\sum_{t\in\text{Fourth class type}} |\theta'(t)|^2 = \frac{1}{2}\sum_{\alpha\in C-Z} (q^2-q)|-\theta(\alpha)-\theta(w\alpha w)|^2 = (q-1)^2(q^2-q)$$

The value of $(\theta'|\theta') = \frac{1}{|G|}[(q-1)^3 + (q-1)^2(q+1) + (q-1)^2(q^2-q) = (q-1)^2(q^2+q)] = 1$.

Theorem 8.2 There are $\frac{1}{2}(q-1)q$ irreducible Cuspidal representations. They are of dimension $q-1$.

Proof. The character table shows that a Cuspidal representation is determined by θ, which is a homomorphism from $K^\times$ to $\mathbb{C}^\times$ such that $\theta(w\alpha w) \neq \theta(\alpha)$. This gives $q(q-1)$ homomorphisms, but since the character table shows that θ and $\theta(w\alpha w)$ generate the same representation, there are really $\frac{1}{2}(q-1)q$ irreducible Cuspidal representations.
The dimension of θ^G is $\frac{|G|}{|C|}(1) = q^2-1$ by the direct sum decomposition of an induced representation. The dimension of $(\mu,\lambda)^G$ is $\frac{|G|}{|ZU|}(1) = q^2-1$ by the direct sum decomposition of an induced representation. So the dimension of a Cuspidal representation is $(q^2-1)-(q^2-q) = q-1$.

9 Conclusion

In this thesis we have found the irreducible characters of $GL_2(\mathbb{F}_q)$. We can summarize the findings in the following table.

type	**number of type**	**dimension**
$\mu \circ \det$	$q-1$	1
$\psi_\mu^G - \mu \circ \det$	$q-1$	q^2-1
ψ^G with $\psi \neq [w]\psi$	$\frac{1}{2}(q-1)(q-2)$	q^2+q
θ' with $\theta \neq [w]\theta$	$\frac{1}{2}(q-1)q$	q^2-q

Table 7: Simple Characters of G

The table shows as many irreducible representations of $GL_2(\mathbb{F}_q)$ as there are conjugacy classes of $GL_2(\mathbb{F}_q)$ so we have found all of the irreducible representations of $GL_2(\mathbb{F}_q)$.

References

[1] Daniel Bump. *Automorphic Forms and Representations.* Cambridge University Press, 1997.
[2] William Fulton, Joe Harris. *Representation Theory.* Springer-Verlag, 1991.
[3] J. A. Green. The Characters of the Finite General Linear Groups. Transactions of the American Mathematical Society (1955), 80(2), 402–447.
[4] Serge Lang. *Algebra.* Springer New York, 2002.
[5] Jean-Pierre Serre. *Linear Representations of Finite Groups.* Springer New York, 1977.