Complex Analysis Qualifying Exam – Fall 2025

Name: ____________________________________

Student ID: ________________________________

Instructions:

No books or notes. You may use without proofs results proved in Conway, Chapters I-XI. However, if using a homework problem, please make sure you reprove it. Present your solutions clearly, with appropriate detail.

You have 180 minutes to complete the test.

Notation: $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
Total		60

Problem 1. [*10 points.*]

How many zeros does the polynomial equation $z^4 - 6z + 3 = 0$ have in the annulus

$$G = \{z \in \mathbb{C} : 1 < |z| < 2\}?$$

Please justify your answer.

Problem 2. [*10 points.*]

Let

$$f : \{z : 0 < |z| < 1\} \to \mathbb{C}$$

be holomorphic and assume that

$$|f(z)| \leq A|z|^{-3/2}$$

for some constant A. Prove that there is a complex constant α such that

$$g(z) := f(z) - \alpha z^{-1}$$

can be extended to a holomorphic function on $\{z : |z| < 1\}$.

Problem 3. [*10 points; 3, 5, 2.*]

Suppose R_1, R_2 are bounded simply connected regions in $\mathbb{C}$. Let $z_1 \in R_1$ and $z_2 \in R_2$.

(i) Prove that there exists a holomorphic bijective function

$$f : R_1 \to R_2$$

such that $f(z_1) = z_2$.

(ii) Suppose that $g : R_1 \to R_2$ is a holomorphic function such that $g(z_1) = z_2$. Prove that

$$|g'(z_1)| \leq |f'(z_1)|.$$

(iii) When does equality occur in (ii)?

Problem 4. [*10 points; 7, 3.*]

Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function in the unit disk. Assume that $|f(z)|$ is constant on each circle $|z| = r$ for $0 < r < 1$; i.e., $|f(re^{i\theta})| = \phi(r)$, for some non-negative function ϕ on $0 < r < 1$.

(i) Assume that $f(0) \neq 0$. Show that f is constant.

(ii) Assume that f has a zero of order $m > 0$ at $z = 0$. Show that $f(z) = cz^m$ for some constant c.

Problem 5. [*10 points.*]

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function, and define $f_n(z) = f(nz)$. Suppose that

$$\mathcal{F} = \{f_n : n \geq 1\}$$

is a normal family on the annulus $\{1 < |z| < 2\}$. Show that f is constant.

Problem 6. [*10 points; 6, 4.*]

(i) Let $G \subset \mathbb{C}$ be a nonempty simply connected region. Show that $G = \mathbb{C}$ if and only if every positive harmonic function $h : G \to \mathbb{R}$ is constant.

(ii) Let $G = \mathbb{C} \setminus \{0\}$. If $h : G \to \mathbb{R}$ is a positive harmonic function, show that h is constant.