### Complex Analysis Qualifying Exam – Fall 2025

Name:				
Student ID:				

#### **Instructions:**

No books or notes. You may use without proofs results proved in Conway, Chapters I-XI. However, if using a homework problem, please make sure you reprove it. Present your solutions clearly, with appropriate detail.

You have 180 minutes to complete the test.

Notation:  $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}.$ 

Question	Score	Maximum	
1		10	
2		10	
3		10	
4		10	
5		10	
6		10	
Total		60	

# Problem 1. [10 points.]

How many zeros does the polynomial equation  $z^4 - 6z + 3 = 0$  have in the annulus

$$G = \{ z \in \mathbb{C} : 1 < |z| < 2 \}?$$

Please justify your answer.

### Problem 2. [10 points.]

Let

$$f:\{z:0<|z|<1\}\to\mathbb{C}$$

be holomorphic and assume that

$$|f(z)| \le A|z|^{-3/2}$$

for some constant A. Prove that there is a complex constant  $\alpha$  such that

$$g(z) := f(z) - \alpha z^{-1}$$

can be extended to a holomorphic function on  $\{z:|z|<1\}$ .

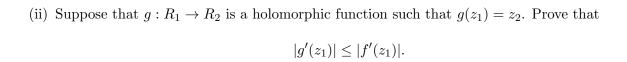
**Problem 3.** [10 points; 3, 5, 2.]

Suppose  $R_1, R_2$  are bounded simply connected regions in  $\mathbb{C}$ . Let  $z_1 \in R_1$  and  $z_2 \in R_2$ .

(i) Prove that there exists a holomorphic bijective function

$$f:R_1\to R_2$$

such that  $f(z_1) = z_2$ .



(iii) When does equality occur in (ii)?

#### **Problem 4.** [10 points; 7, 3.]

Let  $f: \mathbb{D} \to \mathbb{C}$  be a holomorphic function in the unit disk. Assume that |f(z)| is constant on each circle |z| = r for 0 < r < 1; i.e.,  $|f(re^{i\theta})| = \phi(r)$ , for some non-negative function  $\phi$  on 0 < r < 1.

(i) Assume that  $f(0) \neq 0$ . Show that f is constant.

(ii) Assume that f has a zero of order m > 0 at z = 0. Show that  $f(z) = cz^m$  for some constant c.

# Problem 5. [10 points.]

Let  $f: \mathbb{C} \to \mathbb{C}$  be an entire function, and define  $f_n(z) = f(nz)$ . Suppose that

$$\mathcal{F} = \{f_n : n \ge 1\}$$

is a normal family on the annulus  $\{1<|z|<2\}$ . Show that f is constant.

# **Problem 6.** [10 points; 6, 4.]

(i) Let  $G \subset \mathbb{C}$  be a nonempty simply connected region. Show that  $G = \mathbb{C}$  if and only if every positive harmonic function  $h: G \to \mathbb{R}$  is constant.

(ii) Let  $G = \mathbb{C} \setminus \{0\}$ . If  $h: G \to \mathbb{R}$  is a positive harmonic function, show that h is constant.