Name: ____________________

PID: ____________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

1. Write your Name and PID, on the front page of your exam.
2. Read each question carefully, and answer each question completely.
3. Write your solutions clearly in the exam sheet.
4. Show all of your work; no credit will be given for unsupported answers.
5. You may use the result of one part of the problem in the proof of a later part, even if you do not complete the earlier part.
6. You may use major theorems *proved* in class, but not if the whole point of the problem is reproduce the proof of a theorem proved in class or the textbook. Similarly, quote the result of a homework exercise only if the result of the exercise is a fundamental fact and reproducing the result of the exercise is not the main point of the problem.

1. (10 points) Suppose G is a non-cyclic finite group of order pn where p is prime and $n \in \mathbb{Z}^+$. Suppose $\gcd(p!, n) = 1$, and G has an element of order n. Prove that

$$p | \phi(n)$$

where $\phi(n) := |\{k \in \mathbb{Z} | \ 1 \leq k \leq n, \gcd(k, n) = 1\}|$ is the Euler ϕ-function.

2. Suppose G is a finite group, and $\Phi(G)$ is its Frattini subgroup; that means $\Phi(G)$ is the intersection of all maximal subgroups of G. Suppose $G/\Phi(G)$ is nilpotent.

   (a) (4 points) Let P be a Sylow p-subgroup of G. Prove that $P\Phi(G)$ is a normal subgroup of G.

   (b) (5 points) Prove that $P \trianglelefteq G$. (**Hint.** P is a Sylow p-subgroup of $P\Phi(G)$; use Frattini's argument.)

   (c) (1 point) Prove that G is nilpotent.

3. (10 points) Suppose A is a unital commutative ring with no non-zero nilpotent elements. Suppose $N \in M_n(A)$ is nilpotent. Prove that $N^n = 0$. (**Hint.** Prove that $N^n \equiv 0 \pmod{\mathfrak{p}}$ for any $\mathfrak{p} \in \operatorname{Spec}(A)$.)

4. Suppose A is a unital commutative ring, and $A = \langle a_1, \dots, a_m \rangle$; that means the ideal generated by a_i's is A.

   (a) (3 points) Prove that $\langle a_1^{k_1}, \dots, a_m^{k_m} \rangle = A$ for any positive integers k_i's.

   (b) (7 points) Let $S_i := \{1, a_i, a_i^2, \dots\}$ for any $1 \le i \le m$. Prove that

$$\theta : A \to S_1^{-1}A \times \cdots \times S_m^{-1}A, \theta(x) := \left(\frac{x}{1}, \cdots, \frac{x}{1}\right)$$

   is injective.

5. (10 points) Suppose A is a unital commutative ring, and P_1 and P_2 are finitely generated projective A-modules. Prove that $\mathrm{Hom}_A(P_1, P_2)$ is a projective A-module.

6. (a) (7 points) Suppose D is an integral domain and M is a flat D-module. Prove that M is torsion-free.

(b) (3 points) Suppose D is a PID and M is a finitely generated flat A-module. Prove that M is a free D-module.

7. (a) (2 points) Suppose $F/\mathbb{F}_{p^n}$ is a finite field extension where p is prime and $n \in \mathbb{Z}^+$. Prove that $\mathrm{Gal}(F/\mathbb{F}_{p^n}) = \langle \sigma^n \rangle$, where $\sigma(a) := a^p$ is the Frobenius automorphism of F. (You are allowed to use without proof the fact that $F/\mathbb{F}_p$ is a Galois extension and $\mathrm{Gal}(F/\mathbb{F}_p) = \langle \sigma \rangle$.)

(b) (8 points) Suppose $g(x)$ is an irreducible factor of $x^{p^n} - x + 1$ in $\mathbb{F}_{p^n}[x]$ where p is prime and n is a positive integer. Prove that $\deg g = p$. (**Hint.** Suppose α is a zero of $g(x)$ and notice that $\sigma^n(\alpha) = \alpha - 1$.)

8. Suppose $\zeta_n := e^{\frac{2\pi i}{n}} \in \mathbb{C}$, and $K_n := \mathbb{Q}[\zeta_n] \cap \mathbb{R}$.

(a) (3 points) Prove that $K_n/\mathbb{Q}$ is a Galois extension.

(b) (7 points) Suppose $\alpha \in K_n$, $r := \alpha^m \in \mathbb{Q}$ and m is the smallest such positive integer; that means $\alpha^i \notin \mathbb{Q}$ for $1 \leq i < m$. Prove that $m \leq 2$. (**Hint.** Think about the minimal polynomial of α over $\mathbb{Q}$.)

Good Luck!