Algebra qualifying exam Sep 12, 2007

Name: ______________________________

"[n]" means the problem is worth n points.

1. Let G be a group of order $240 = 2^4 \cdot 3 \cdot 5$.
a [10]. How many p-Sylow subgroups might G have, for $p = 2, 3, 5$?

b [15].

If G has a subgroup of order 15, show that it has an element of order 15.

c [15]. Say G *doesn't* have a subgroup of order 15.
Show that the number of 3-Sylows is 10 or 40.

2. Let R denote a commutative ring and I an ideal, $I \neq R$. We say that R has nilpotents if $\exists r \in R, n \in \mathbb{N}, r \neq 0, r^n = 0$.

a [10]. Give an example where R/I has nilpotents but R doesn't.

b [10]. Give an example where R has nilpotents but R/I doesn't.

3. Let $\phi : \mathbb{C}[x] \to F$ be a ring homomorphism where F is a field, $\phi(1) \neq 0$.
a [10]. Give an example where ϕ is not onto.

b [20]. If ϕ is onto, show that $F \cong \mathbb{C}$.

4a [10]. Give an example of two finitely generated $\mathbb{Z}$-modules, M and N, such that M, N are not isomorphic (as $\mathbb{Z}$-modules) but $\mathbb{Q} \otimes_{\mathbb{Z}} M \cong \mathbb{Q} \otimes_{\mathbb{Z}} N$ (as $\mathbb{Q}$-modules).

4b [10]. Let M be a finitely generated $\mathbb{R}[x]$-module, described using the classification of f.g. modules over a PID. Give a similar description of $\mathbb{C}[x] \otimes_{\mathbb{R}[x]} M$ as a $\mathbb{C}[x]$-module.

4c [10]. Show that if M, N are two finitely generated $\mathbb{R}[x]$-modules, and
$\mathbb{C}[x] \otimes_{\mathbb{R}[x]} M \cong \mathbb{C}[x] \otimes_{\mathbb{R}[x]} N$ (as $\mathbb{C}[x]$-modules),
then $M \cong N$ (as $\mathbb{R}[x]$-modules).

5 [15]. Let F be a field of characteristic p, and $f \in F[x]$ a polynomial, $f(x) = \sum_i f_i x^i$. Give necessary and sufficient conditions on the $\{f_i\}$ for $f(x^p)$ itself to be a pth power, i.e. $\exists g(x)$ s.t. $f(x^p) = g(x)^p$. In particular, prove that your condition is necessary.

6. Let $F \geq K$ be an extension field of degree 2.

a [10]. If K is characteristic not 2, show F is Galois over K.

b [5]. Give an example where F is Galois over K even though char $K = 2$.

c [10]. Give an example where F is not Galois over K.