Department of Mathematics
MA/PhD Qualifying Examination
in Algebra

Examiners: Philip Gill and Lance Small

9:00am–12 Noon, AP&M 7421
Monday May 23, 2005

NAME ____________________

#1	20	
#2	20	
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#12	20	
#13	20	
Total	260	

- Do all problems.
- For grading purposes. separate your answers to 1–3 from your answers to 4–13.
- Add your name in the box provided and staple this page to your solutions.

Question 1. Let $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be the transformation such that

$$T(X) = \frac{1}{2}(X - X^T).$$

(a) Prove that T is a linear transformation.

(b) Determine the null space of T and find its dimension.

(c) Derive the matrix representation of T in terms of the standard basis for M_3.

Question 2. Prove that a triangular matrix is normal if and only if it is diagonal.

Question 3. Assume that (λ, x) is an eigenpair of $A \in \mathbb{C}^{n\times n}$ such that $\mathrm{am}(\lambda) = \mathrm{gm}(\lambda) = 1$. Prove that there exists a nonsingular matrix $(x \quad X)$ with inverse $(y \quad Y)^*$ such that

$$\begin{pmatrix} y^* \\ Y^* \end{pmatrix} A (x \quad X) = \begin{pmatrix} \lambda & 0 \\ 0 & M \end{pmatrix}.$$

Question 4. Let G be a finite abelian group of order n. Suppose that G has a unique subgroup of order d for each positive divisor of n. Prove that G is cyclic.

Question 5. Prove that a group of order 120 is not simple.

Question 6. Let G be a group whose center has index n. Show that every conjugacy class in G has at most n elements.

Question 7. Let F be a prime field(the rationals or a field with p elements). Prove that the algebraic closure of F is infinite dimensional over F.

Question 8. Let F be an infinite field. If L is a finite dimensional extension field of F and there are only finitely many intermediate fields, show that $L = F(u)$, for some u.

Question 9. Let $p(x) = x^3 + 3x^2 + 2$. Find the Galois group of the splitting field $p(x)$ over the rationals and over the field of five elements.

Question 10. Classify all rings with identity elements that have nine elements.

Question 11. Let R be a commutative ring with identity element. Suppose that for each $x \in R$ there is an $n(x) > 1$ such that $x^{n(x)} = x$. Show that every prime ideal of R is maximal.

Question 12. State and prove the Hilbert Basis Theorem.

Question 13. Show that there are infinitely many maximal right ideals in n x n matrices over the rationals when $n > 1$