Algebra Qualifying Exam - Spring 2012

Instructions:

Please print your name. Read each question carefully, and show all your work. You may quote major theorems we proved in class or which are proved in the text, unless the point of the problem is to reproduce the proof of such a theorem. However, make as clear as you can what result you are quoting. Generally, avoid quoting the results of homework exercises.

When a problem has multiple parts, you may use the conclusion from part (a) in your proof of part (b), even if you do not complete part (a).

Problem 1. (15 pts)

Let G be a group of order 2012, which has prime factorization $2^2 \cdot 503$.

(a). (10pts) Classify all such groups G in terms of semidirect products of known groups. (Explain what the possible semidirect products are, but you do not need to decide which of them are isomorphic.)

(b). (5 pts) Show that any group of order 2012 has a non-trivial center.

Problem 2. (15 pts)

Let R be a UFD. Let S be any multiplicative system of nonzero elements in R, and let $S^{-1}R$ be the localization of R at S.

(a). (8 pts) Let x be an irreducible element of R. Show that the element $\frac{x}{1}$ is either irreducible in $S^{-1}R$ or a unit. Conversely, show that every irreducible in $S^{-1}R$ is an associate of $\frac{x}{1}$ for some x which is irreducible in R.

(b). (7 pts) Show that (i) every element of $S^{-1}R$ is a finite product of irreducibles; and (ii) Every irreducible element of $S^{-1}R$ is prime. (It is well-known that these conditions imply that $S^{-1}R$ is a UFD, so you have proved that $S^{-1}R$ is also a UFD.)

Problem 3.(10 pts)

Let R be a commutative ring.

(a). (5 pts) Suppose that R is noetherian. Prove that if $\phi : R \to R$ is a surjective ring homomorphism, then ϕ is injective.

(b). (5 pts) If R is not noetherian, must a surjective ring homomorphism $\phi : R \to R$ be injective? Prove or give a counterexample.

Problem 4. (15 pts)

Let R be a commutative ring. Recall that an R-module Q is *flat* if $- \otimes_R Q$ is an exact functor; namely, given any short exact sequence of R-modules $0 \to M \to N \to P \to 0$, the sequence

$$0 \to M \otimes_R Q \to N \otimes_R Q \to P \otimes_R Q \to 0$$

remains exact.

(a). (10 pts) Let Q be any module over an integral domain R. Show that if Q is flat, then Q is torsionfree.

(b). (5 pts) In terms of the classification theorem for modules over PIDs, characterize exactly which finitely generated modules over a PID R are flat.

Problem 5. (15 pts)

Consider a polynomial $f \in K[x]$, where K is an algebraically closed field. Suppose that f has the property that for all matrices $A \in M_n(K)$ of any size n, if $f(A) = 0$, then A is a diagonalizable matrix; then we say that the polynomial f *forces diagonalizibility.*

(a). (10 pts) Characterize by a simple rule exactly which polynomials in $K[x]$ force diagonalizibility. Prove your answer.

(b). (5 pts) Fix $m \geq 1$. Is every square matrix A with entries in K satisfying $A^m = I$ diagonalizable? (The answer depends on K.)

Problem 6. (15 pts)

Let K be the splitting field over $\mathbb{Q}$ of the polynomial $f(x) = (x^3 - 2)(x^2 - 3)$.

(a). (5 pts) Show that $[K : \mathbb{Q}] = 12$.

(b). (5 pts) Take as given that every non-Abelian group of order twelve is isomorphic to either A_4, the dihedral group D_{12}, or else the semidirect product $\mathbb{Z}/3\mathbb{Z} \rtimes_\phi \mathbb{Z}/4\mathbb{Z}$ where ϕ sends the generator of $\mathbb{Z}/4\mathbb{Z}$ to the non-identity automorphism of $\mathbb{Z}/3\mathbb{Z}$. For each of these three groups, calculate what a Sylow-2 subgroup is up to isomorphism, and whether or not it is normal in the group.

(c). (5 pts) Which group of order twelve must the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ be?

Problem 7. (10 pts)

Suppose that K is a finite field with $|K| = q$. Show that if $f \in K[x]$ is irreducible, then f divides $x^{q^n} - x$ in $K[x]$ if and only if $\deg f$ divides n.