Name: ____________________

PID: ____________________

Question	Points	Score
1	5	
2	8	
3	8	
4	5	
5	5	
6	6	
7	8	
Total:	45	

1. Write your name on the front page of your exam.
2. Read each question carefully, and answer each question completely.
3. Write your solutions clearly in the exam sheet.
4. Show all of your work; no credit will be given for unsupported answers.
5. You may use the result of one part of the problem to prove a later part, even if you have not completed the earlier part.

1. (5 points) Suppose $p < q < \ell$ are distinct primes. Prove that there is no simple group of order $pq\ell$.

2. Suppose N is a normal subgroup of G and P is a Sylow p-subgroup of N.

(a) (2 points) Prove that there exists $Q \in \mathrm{Syl}_p(G)$ such that $P = Q \cap N$.

(b) (2 points) Prove that $N_G(Q) \subseteq N_G(P)$.

(c) (4 points) Prove that $g \in \langle N_G(P) \cup gN_G(P)g^{-1} \rangle$ for every $g \in G$.

3. Suppose F is a field. Let $m \leq n$ be two positive integers and

$$M_{m,n} := F[x]/\langle x^n \rangle \otimes_F F[x]/\langle x^m \rangle.$$

Notice that $M_{m,n}$ is an $F[x]$-module where

$$x \cdot (\overline{a(x)} \otimes \overline{b(x)}) = \overline{xa(x)} \otimes \overline{xb(x)}$$

and $\overline{\bullet}$ denotes the corresponding coset. (Let's emphasize that the tensor is over F, and not over $F[x]$.)

(a) (2 points) Prove that $\operatorname{Ann}(M_{m,n}) = \langle x^m \rangle$.

(b) (3 points) Let $d(M_{m,n})$ be the minimum number of generators of $M_{m,n}$ as an $F[x]$-module. Prove that $d(M_{m,n}) = \dim_F(M_{m,n}/xM_{m,n}) = m+n-1$.

(This question has three parts.)

(c) (3 points) Find the multiplicity of x^m among the invariant factors of $M_{m,n}$.

4. (5 points) Let A be a unital commutative ring. Suppose M and N are two finitely generated projective A-modules. Prove that $\mathrm{Hom}_A(M, N)$ is a projective A-module.

5. (5 points) Suppose A is a unital commutative ring and $\langle a_1, \ldots, a_n \rangle = A$. Suppose M is an A-module and $S_{a_i} := \{1, a_i, a_i^2, \ldots\}$. Suppose that $S_{a_i}^{-1} M = 0$ for all i. Prove that $M = 0$.

(**Hint.** For $x \in M$, consider the annihilator $\operatorname{ann}(x)$ of x.)

6. (6 points) Suppose p is an odd prime. Let E be a splitting field of $x^p - 2x - 1$ over $\mathbb{F}_p$. Prove that

$$\mathrm{Gal}(E/\mathbb{F}_p) \simeq \langle 2 + p\mathbb{Z} \rangle \subseteq (\mathbb{Z}/p\mathbb{Z})^\times.$$

7. Let p be prime and F_0 a field of characteristic zero. Suppose F_0 satisfies the following property.

$$\text{If } K/F_0 \text{ is a finite field extension and } K \neq F_0, \text{ then } p \mid [K : F_0].$$

(a) (4 points) Prove that for every finite Galois extension K/F_0, the Galois group $\mathrm{Gal}(K/F_0)$ is a p-group.
(**Hint.** Consider a Sylow p-subgroup of $\mathrm{Gal}(K/F_0)$.)

(b) (4 points) Suppose F_1/F_0 is a field extension, $[F_1 : F_0] = p$, and F_1 has no field extension of degree p. Prove that F_1 is algebraically closed.

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Good Luck!