Name: ______________________________ S.I.D.: ______________________________

Qualifier Exam in Applied Algebra

May 13, 2019

	Full	Real
# 1	10	
# 2	10	
# 3	10	
# 4	10	
# 5	10	
# 6	10	
# 7	10	
# 8	10	
# 9	10	
# 10	10	
Total	100	

Notes: 1) For computational questions, no credit will be given for unsupported answers gotten directly from a calculator. 2) For proof question, no credit will be given for no reasons or wrong reasons.

1.

2.

3.

4.

5. (10 points) Let $G = GL_2(\mathbb{R})$ be the group of invertible 2×2 real matrices and let $X, Y : G \to GL_d(\mathbb{C})$ be two complex matrix representations of G with the same degree d. If X and Y have the same character, are X and Y necessarily isomorphic? Justify your answer.

6. (5 + 5 points) Let $D_4 = \langle r, s \mid r^4 = s^2 = 1,\ srs = r^{-1} \rangle$ be the dihedral group of symmetries of the square.

(a) Write down the character table of D_4.

(b) Let V be the 2-dimensional 'defining' D_4-module obtained by centering the square at the origin in the plane and extending symmetries of the square to linear transformations of the plane. Give the tensor product $V \otimes V$ the structure of a D_4-module by setting

$$g.(v \otimes v') := (g.v) \otimes (g.v')$$

for all $g \in D_4$ and $v, v' \in V$. Calculate the decomposition of $V \otimes V$ into irreducible D_4-modules.

7. (5 + 5 points) Let G be a finite group and let $X : G \to GL_d(\mathbb{R})$ be an irreducible matrix representation of G over the field of real numbers. Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be an endomorphism of X.

(a) Give an example to show that T is not necessarily a scalar transformation.

(b) Suppose that T is not a scalar transformation. Consider the representation $X' : G \to GL_d(\mathbb{C})$ given by viewing real matrices as complex matrices:

$$X'(g) := X(g) \quad \text{for all } g \in G.$$

Is it possible for X' to be irreducible (as a complex matrix representation)? Justify your answer.

8. (5 + 5 points) Let S_n be the symmetric group on n letters.

(a) Calculate the character table of the product group $S_3 \times S_2$.

(b) Let $\lambda = (3,2) \vdash 5$ and let S^λ be the associated irreducible representation of S_5. Calculate the decomposition of the restricted module $S^\lambda \downarrow_{S_3 \times S_2}^{S_5}$ into irreducibles.

9. (10 points) Let $\mathbb{A}$ be a finite-dimensional algebra over $\mathbb{C}$ with center $\mathcal{Z}(\mathbb{A})$, and let (V, ρ) be an irreducible representation of $\mathbb{A}$. Show that $\rho(z) = \chi(z)I_V$ for each $z \in \mathcal{Z}(\mathbb{A})$, where $\chi(z)$ is a scalar. Show that the map $\chi\colon \mathcal{Z}(\mathbb{A}) \to \mathbb{C}$ defined by $z \mapsto \chi(z)$ is an algebra homomorphism.

10. (10 points) Let $\mathbb{A}$ be a finite-dimensional algebra over $\mathbb{C}$ and let (V, ρ) be a finite-dimensional representation of $\mathbb{A}$. Show that the isotypic decomposition of (V, ρ) is multiplicity free if and only if $\mathrm{End}_{\mathbb{A}} V$ is commutative.