Applied Algebra Qualifying Exam: Part I

9:00am–Noon, AP&M 6402
Tuesday May 28th, 2013

Name ____________________

#1	20	
#2	20	
#3	20	
#4	20	
Total	80	

- Do all four problems.
- This part of the exam will represent 40% of your total score.
- Add your name in the box provided and staple this page to your solutions.
- Notation:
  - $\mathcal{M}_{m,n}$ denotes the set of $m \times n$ matrices with complex entries.
  - If $m = n$, $\mathcal{M}_{m,n}$ is denoted by $\mathcal{M}_n$.
  - $\mathbb{C}^n$ is the set of column vectors with n complex entries.
  - x^H is the Hermitian transpose of a vector or matrix x.
  - $\text{eig}(A)$ is the set of eigenvalues of the matrix A (counting multiplicities).
  - $\text{Re}(\lambda)$ and $\text{Im}(\lambda)$ denote the real and imaginary parts of the scalar λ.

Question 1.

(a) (8 points) Prove the Schur decomposition theorem for a matrix $A \in M_n$.

(b) (12 points) Prove that for A, $B \in \mathcal{M}_n$, if $x^H A x = x^H B x$ for all $x \in \mathbb{C}^n$, then $A = B$.

Question 2.

(a) (10 points) Prove that every $A \in \mathcal{M}_n$ may be written *uniquely* as $A = S + iT$, where S and T are *Hermitian*.

(b) (10 points) For any $A \in \mathcal{M}_n$, consider the unique expansion $A = S + iT$, where S and T are Hermitian. Prove that for any $\lambda \in \text{eig}(A)$, it holds that

$$\lambda_n(S) \leq \text{Re}(\lambda) \leq \lambda_1(S) \quad \text{and} \quad \lambda_n(T) \leq \text{Im}(\lambda) \leq \lambda_1(T),$$

where, by convention, the eigenvalues of a Hermitian matrix $C \in \mathcal{M}_n$ are arranged in nonincreasing order, i.e.,

$$\lambda_1(C) \geq \lambda_2(C) \geq \cdots \geq \lambda_n(C).$$

Question 3.

(a) (4 points.) Define the p-norm $\|A\|_p$ and Frobenius norm $\|A\|_F$ of a matrix $A \in \mathcal{M}_{m,n}$.

(b) (10 points) Suppose that $D \in \mathcal{M}_n$ with $D = \text{diag}(d_1, d_2, \ldots, d_n)$. Prove that for all $1 \leq p \leq \infty$ the p-norm of D is given by $\|D\|_p = \max_{1 \leq i \leq n} |d_i|$.

(c) (6 points) Given $b \in \mathbb{C}^{n-1}$, find $\|B\|_2$ for the matrices

$$B = \begin{pmatrix} 0 & b^H \\ b & 0 \end{pmatrix} \quad \text{and} \quad B = bb^H.$$

(Show your work. Simply writing down the answer will not be sufficient.)

Question 4.

(a) (15 points) Prove that if $A \in \mathcal{M}_n$ is positive semidefinite, then there exists a *unique* positive semidefinite X such that $A = X^2$.

(b) (5 points) Let X be a matrix whose columns define a basis for a subspace $\mathcal{X} \subset \mathbb{C}^n$. Consider the matrix $\widehat{X} = X|X|^{-1}$, where $|X|$ denotes the modulus of X, i.e., $|X| = (X^H X)^{\frac{1}{2}}$. Prove that $\widehat{X}$ exists and that $\widehat{X}\widehat{X}^H$ is an orthogonal projection onto $\mathcal{X}$.

Applied Algebra Qualifying Exam: Part II
May 28, 2013

Do as many problems as you can, but you must attempt at least 5 problems where two of the problems are from problems 1-5, one problem for 6-7, and one problem are from problems 8-9. The point values are relative values for this part of the exam. Your final score will be scaled so that this part of the exam will represent 60% of your point total.

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, $\mathbb{Q}$ equal the rationals and $\mathbb{C}$ denote the complex numbers.
Suppose that $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k)$ is a partition of n. Then A^λ denotes the irreducible representation of the symmetric group S_n such that the Frobenius image of $\chi^{A^\lambda} = \chi^\lambda$ is the Schur function $S_\lambda(x_1, \ldots, x_N)$ where $N > n$ and $S_{\lambda_1} \times \cdots \times S_{\lambda_k}$ denotes the Young subgroup of S_n corresponding to λ.

(1)(30 pts.) Let H be a subgroup of G and $A : H \to GL_n(\mathbb{C})$ be a representation of H. Let $\chi^A : H \to \mathbb{C}$ be the character of A. Define $\chi^{\overline{A}} : G \to \mathbb{C}$ by

$$\chi^{\overline{A}}(\sigma) = \begin{cases} \chi^A(\sigma) & \text{if } \sigma \in H \text{ and} \\ 0 & \sigma \in G - H. \end{cases}$$

(a) Define the representation $A \uparrow_H^G$.

(b) Prove that $\chi^{A \uparrow_H^G} = \frac{1}{|H|} \sum_{\sigma \in G} \sigma \cdot \chi^{\overline{A}} \cdot \sigma^{-1}$.

(c) State and prove the Frobenius Reciprocity Theorem.

(2) (40 pts)
(a) Compute the values of the character $\chi^{(1,2^2)}$ on the conjugacy classes of S_5.

(b) Find the character table of $S_3 \times S_2$.

(c) Decompose the $A^{(1,2^2)} \downarrow_{S_3 \times S_2}^{S_5}$ as a sum of irreducible characters of $S_3 \times S_2$.

(3) (40 pts) Let Q be the quaternion group of order 8 defined by the relations

$$a^4 = 1,\ a^2 = b^2,\ \text{and } b^{-1}ab = a^3.$$

(a) Show that $ba = ab^3 = a^3 b$ and, hence, that every element of Q is of the form a^i or $a^i b$ for some $i \in \{0, 1, 2, 3\}$.

(b) Verify that the conjugacy classes of G are $C_1 = \{1\}$, $C_2 = \{a^2\}$, $C_3\{a, a^3\}$, $C_4 = \{b, a^2 b\}$, and $C_5 = \{ab, a^3 b\}$.

c) Show that $H = \{1, a^2\}$ is a normal subgroup of G for which G/H is isomorphic to $Z_2 \times Z_2$.

(d) Give the character character table for the lifting of the four linear characters of Q/H to Q.

(e) Use parts (c) and (d) to give the complete character table for Q.

(4) (30 pts)

(a) Let T denote the trivial representation on the Young subgroup $S_2 \times S_3 \times S_1$ of S_6 and Alt denote the alternating representation on the Young subgroup $S_2 \times S_3 \times S_1$ of S_6. Express the characters of

$$T \uparrow_{S_2 \times S_3 \times S_1}^{S_6} \text{ and } Alt \uparrow_{S_2 \times S_3 \times S_1}^{S_6} .$$

as a sum of irreducible characters of S_6.

(b) Find the decomposition of the Kronecker product $A^{(1,4)} \otimes A^{(1,2^2)}$ as a sum of irreducible representations of S_5.

(c) Find the decomposition of $A^{(1,2)} \times A^{(1,3)} \uparrow_{S_3 \times S_4}^{S^7}$ as a sum of irreducible representations of S_7.

(5) (40 pts.) Let G and H be finite groups and let $A : G \rightarrow GL_n(C)$ and $B : H \rightarrow GL_m(C)$ be representations of G and H respectively.

a) Show that $A \times B : G \times H \rightarrow GL_{nm}(C)$ is representation where for $(\sigma, \tau) \in G \times H$,

$$A \times B((\sigma, \tau)) = A(\sigma) \otimes B(\tau)$$

and for matrices M and N, $M \otimes N$ is the Kronecker product of M and N.

b) Show that $A \times B$ is an irreducible representation of $G \times H$ if and only if A is an irreducible representation of G and B is an irreducible representation of H.

c) Show that every irreducible representation of $G \times H$ is of the form $A \times B$ where A is an irreducible representation of G and B is an irreducible representation of H.

(d) Show that it is not always the case that if C is a representation of $G \times H$, then C is similar to a representation of the form $A \times B : G \times H \rightarrow GL_n(\mathbb{C})$ where A is representation of G and B is representation of H. (Hint: Consider the two dimensional representations of $S_2 \times S_2$.)

(6) (40 pts.) Consider the equations

$$\begin{aligned} x^2 - xy - 2x &= 0 \\ y^2 - 2xy - y &= 0 \end{aligned}$$

(a) Let I be the ideal of $\mathbf{C}[x, y]$ generated by these equations. Find the reduced Groebner basis for I relative to lexicographic order where $y > x$.

(b) Find a reduced Groebner basis for $\mathbf{C}[x] \cap I$.

(c) Find all solutions to these equations that lie $\mathbf{C}^2$.

(d) Find a vector space basis for $\mathbf{C}[x, y]/I$.

(7) (30 pts.) Let S be the parametric surface defined by

$$\begin{aligned} x &= u - 2v \\ y &= uv \\ z &= v \end{aligned}$$

(a) Compute a reduced Groebner basis for the ideal generated by this set of equations relative to the lexicographic order where $u > v > x > y > z$.

(b) Find the equation of the smallest variety V that contains S.

(c) Show that $S = V$.

(8) (40 pts.) Let k be an algebraically closed field.
Two ideals I and J of $k[x_1, \ldots, x_n]$ are said to be *comaximal* if and only if $I + J = k[x_1, \ldots, x_n]$.

(a) State the Weak Nullstellenszat and Hilbert's Nullstellensatz Theorem.

(b) Show that two ideals I and J are comaximal if and only if $V(I) \cap V(J) = \emptyset$.

(c) Show that if I and J are ideals in $k[x_1, \ldots, x_n]$, then $I \cap J = (tI + (1-t)J) \cap k[x_1, \ldots, x_n]$

(d) Show that if $I = \langle f \rangle$ and $J = \langle f \rangle$, then $I \cap J = \langle h \rangle$ where h is a least common multiple of f and g.

(9) (30 pts.) Let $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$.

(a) Show that A generates a matrix group G of order three.

(b) Find a set of homogeneous G-invariant polynomials which generate $\mathbb{C}[x, y]^G$.

(c) Compute the Hilbert Series of $\mathbb{C}[x, y]^G$.