Complex Analysis Qualifying Exam – Spring 2017

Name: ______________________________

Student ID: ___________________________

Instructions:

You have 3 hours. No textbooks and notes are allowed. Solve 7 of the following 8 questions.

Notation: $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
Total		70

Problem 1. [*10 points.*]

Using the calculus of residues, compute

$$\int_0^\infty \frac{\cos x}{(1+x^2)^2}\, dx.$$

Problem 2. [*10 points.*]

Let $f : \mathbb{C} \to \mathbb{C}$ be entire. Assume that the function $g(z) = f(z) \cdot f\left(\frac{1}{z}\right)$ is bounded on $\mathbb{C} \setminus \{0\}$. Show that $f(z) = cz^m$.

Problem 3. [*10 points.*]

Let $n \geq 0$ be an integer. Assume $f : \mathbb{C} \to \mathbb{C}$ is an entire function such that

$$|f(z)| \leq 1 + (\log(1 + |z|))^n .$$

Prove that f is constant.

Problem 4. [*10 points; 5, 5.*]

(i) Construct an entire function with simple zeros at $\left\{\sqrt{n} + \frac{1}{\sqrt{n}} : n = 1, 2, \ldots\right\}$ and no other zeroes.

(ii) Construct a meromorphic function with simple poles at $z = n\sqrt{n}$ and residues equal to $\sqrt{n}$ for $n = 1, 2, \ldots$.

Problem 5. [*10 points.*]

Let $\mathcal{F}$ be the family of holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ defined over the open unit disc with

$$f(0) = 1 \text{ and } \operatorname{Re} f > 0.$$

Show that $\mathcal{F}$ is a normal family.

Problem 6. [*10 points.*]

Find the number of roots of the polynomial

$$z^{87} + 36z^{57} + 71z^4 + z^3 - z + 1$$

in the region $1 \leq |z| \leq 2$.

Problem 7. [*10 points.*]

Show that the function $f(z) = \cos\left(\sqrt{z}\right)$ is entire. Determine the order, rank and genus of f.

Problem 8. [*10 points; 5, 5.*]

(i) Let $u : \mathbb{C} \to \mathbb{R}$ be a harmonic function which is bounded. Show that u is constant.

(ii) Let $H = \{z : \text{ Im } z > 0\}$ denote the upper half plane. Let $u : \overline{H} \to \mathbb{R}$ be a continuous bounded function which is harmonic in H and $u = 0$ on ∂H. Show that u is constant.