COMPLEX ANALYSIS, MATH 220

Qualifying Exam, May 29, 2018

Instructions: 3 hours. All problems are worth an equal number of points though they are not of equal difficulty. You may use without proof results proved in Conway up to and including Chapter X.2. When using a result from the text, be sure to explicitly verify all hypotheses in it. Present your solutions clearly, with appropriate detail.

Notation and terminology: The unit disk is denoted by $\mathbb{D}$. G is a region, i.e., an open and connected subset of $\mathbb{C}$. The space of analytic functions in G is denoted by $H(G)$.

1. Let f be an analytic function on $\mathbb{C}$ that satisfies the inequality

$$|f(z)| \leq e^{\operatorname{Re} z}$$

for all $z \in \mathbb{C}$. Prove that either $f(z) = 0$ for all $z \in \mathbb{C}$ or $f(z) \neq 0$ for all $z \in \mathbb{C}$.

2. Use the method of residues to evaluate

$$\int_0^\infty \frac{dt}{t^3+1}.$$

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3. Show that if $G \neq \mathbb{C}$ is a simply connected subset of $\mathbb{C}$, $f : G \to G$ is analytic, and $f(z)$ is not identically equal to z, then f has at most one fixed point in G.

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4. Let u_n be a sequence of harmonic functions in $\mathbb{D}$ that are continuous in $\overline{\mathbb{D}}$. Assume that u_n converges uniformly on $\partial\mathbb{D}$ to a function f. Show that u_n converges in the space of harmonic functions in $\mathbb{D}$ to a harmonic function u that is continuous in $\overline{\mathbb{D}}$ and equal to f on $\partial\mathbb{D}$.

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5. Let $\alpha_n := 1 - \frac{1}{n^2}$. Construct a function $f \in H(\mathbb{D})$ whose sequence of zeros (counting multiplicity) is precisely $\{\alpha_n\}$ and for which $|f(z)| \leq 1$ in $\mathbb{D}$.
Hint: $-\varphi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}$ has a simple zero at $z = \alpha$ and $|\varphi_\alpha(z)| \leq 1$ in $\mathbb{D}$.

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6. Let $\{f_n\}$ be a sequence in $H(G)$ such that $f_n(z)$ converges for every $z \in G$ to a complex value $f(z)$, i.e., $\lim_{n\to\infty} f_n(z) = f(z) \in \mathbb{C}$. Assume that $|f_n'(z)| \leq h(z)$ for some continuous real-valued function h in G. Show that $f(z)$ is analytic in G and $f_n \to f$ in $H(G)$.
