Complex Analysis Qualifying Exam – Spring 2020

Name: ______________________________

Student ID: ___________________________

Instructions:

You do not have to reprove any results from Conway or shown in class. However, if using a homework problem, please make sure you reprove it.

You have 180 minutes to complete the test.

Notation: $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
Total		70

Problem 1. [*10 points.*]

Let $f(z) = \pi^2 z^5 e^{-2z} - 1$. How many roots does f have in $\mathbb{D}$? How many simple roots does f have in $\mathbb{D}$?

Hint: $e < \pi$.

Problem 2. [*10 points.*]

Prove or disprove the following statement.

Let $U = \{z \in \mathbb{C} : |z| > 3\}$. *There exists a holomorphic function* f *in* U *such that*

$$f'(z) = \frac{z^2 + 2}{z(z-1)(z-2)}.$$

Problem 3. [*10 points.*]

For $a \in (-1, 1)$, let $D_a = \{z \colon |z| < 1,\ \operatorname{Im} z > a\}$. For each such a, either find a Möbius transformation of D_a onto the quadrant $Q = \{w = re^{i\theta} \colon r > 0,\ 0 < \theta < \frac{\pi}{2}\}$, or show that such a transformation cannot exist.

Problem 4. [*7, 3 points.*]

Let $G \subset \mathbb{C}$ be a proper open connected subset. Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ denote the extended plane.

(i) Assume $\mathbb{C}_\infty - G$ is connnected. Prove the following statement.

Let f be an analytic function from G to G which is not the identity map. Then f has at most one fixed point in G.

(ii) Is the statement still true if $\mathbb{C}_\infty - G$ is not connected? If true, give a proof. If false, give a counterexample.

Problem 5. [*10 points.*]

Let $A_1 = \mathbb{D} - \left\{0, \frac{1}{2}\right\}$ and $A_2 = \mathbb{D} - \left\{0, -\frac{1}{2}\right\}$. Find all bijective analytic maps from A_1 to A_2.

Problem 6. [*10 points.*]

Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $|f'(z)| \leq e^{|z|}$ and

$$f\left(\sqrt{n}\right) = 0$$

for all positive integers $n > 0$. Show that $f = 0$.

Problem 7. [*10 points.*]

Let $\mathcal{H}$ be the family of harmonic functions $h : \mathbb{D} \to \mathbb{R}$ with $h(0) = 1$ and $h(z) > 0$ for all $z \in \mathbb{D}$.

Show that every sequence in $\mathcal{H}$ admits a subsequence that converges uniformly on compact subsets of $\mathbb{D}$ to a function in $\mathcal{H}$.