MATH 220: Complex Analysis
Qualifying Exam. September 10, 2007

General instructions: 3 hours. No books or notes. Be sure to motivate all (nontrivial) claims and statements. You may use without proof any result proved in the text (Conway up to Ch X). You need to reprove any result given as an exercise.

Notation: G denotes an open region in $\mathbb{C}$, $\mathcal{O}(G)$ denotes the space of analytic functions in G, and $\mathrm{Har}(G)$ denotes the space of harmonic functions in G. $B(a,r)$ denotes the disk $\{z\colon |z-a|<r\}$. $\mathbb{D}$ denotes the unit disk $B(0,1)$. $f^{(n)}(z)$ denotes the nth derivative of $f(z)$.

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1. (50p) Determine if the statements below are **True** or **False**. If **True**, give a brief proof. If **False**, give a counterexample (or prove your assertion in another way, if you prefer). If you claim an assertion follows from a theorem in the text, name the theorem (or describe it otherwise) and explain carefully how the conclusion follows.

(a) (10p) Let (X,d) be a metric space. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence and assume that every subsequence has a convergent subsequence. Then, $\{x_n\}_{n=1}^{\infty}$ is convergent.
(b) (10p) Let $\gamma_+(t)=e^{it}$, $t\in[0,\pi]$, and $\gamma_-(t)=e^{-it}$, $t\in[0,\pi]$. Then, we have

$$\int_{\gamma_+} e^{1/z^2}\,dz=\int_{\gamma_-} e^{1/z^2}\,dz.$$

(c) (10p) If $f(z)$ is an entire function and there are $a\in\mathbb{C}$, $r>0$, such that $f(\mathbb{C})\subset\mathbb{C}\setminus\overline{B(a,r)}$, then $f(z)$ is constant.
(d) (10p) There is an analytic bijection (biholomorphism) from $\mathbb{D}\setminus\{0\}$ onto $\mathbb{C}\setminus\{0\}$.
(e) (10p) The function $u(z):=\ln|z^2+1|$ is harmonic in $\mathbb{C}\setminus\{-i,i\}$.

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2. (30p) Let $f(z)$ be analytic in $\mathbb{D}$ and assume $|f(z)|\le 1$ for all $z\in\mathbb{D}$.
(a) (15p) Suppose that $f(z)$ has a zero of order m at $z=0$. Show that

$$|f(z)|\le|z|^m.$$

(b) (15p) Suppose that $f(0)=\alpha$ and $f'(0)=0$. Show that, for $|z|\le\sqrt{|\alpha|}$,

$$|f(z)|\ge\frac{|\alpha|-|z|^2}{1+|\alpha||z|^2}.$$
