Complex Analysis Qualifying Exam – Fall 2016

Name: ______________________________

Student ID: ______________________________

Instructions:

You have 3 hours. No textbooks and notes are allowed. Make sure to state clearly the hypotheses of any results used.

Solve at least 7 of the following 8 problems. You have 180 minutes to complete the test.

Notation: $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
Total		80

Problem 1. [*10 points.*]

Prove that if f is an entire function such that $\lim_{z \to \infty} f(z) = \infty$, then f must be a polynomial.

Problem 2. [*10 points.*]

Assume that $f : \mathbb{D} \to \mathbb{D}$ is an analytic function such that $f(0) = 0$. Show that

$$g(z) = \sum_{n=0}^{\infty} f(z^n)$$

converges to an analytic function on $\mathbb{D}$.

Problem 3. [*10 points.*]

Using the calculus of residues, compute

$$\int_0^\infty \frac{\log x}{x^2+1}\,dx.$$

Problem 4. [*10 points.*]

Let $f : \overline{\mathbb{D}} \to \mathbb{C}$ be a continuous function which is analytic on $\mathbb{D}$. Assume that there exists $0 < \alpha \leq 2\pi$ such that $f(e^{i\theta}) = 0$, for all $\theta \in (0, \alpha)$. Prove that $f(z) = 0$, for every $z \in \mathbb{D}$.

Problem 5. [*10 points.*]

Let $U \subset \mathbb{C}$ be a connected open and let $a \in U$. Let $f_n : U \to \mathbb{D}$ be a sequence of analytic functions such that $f_n(a) = 0$, for all $n \geq 1$. Prove that there exists an analytic function $f : U \to \mathbb{D}$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which converges uniformly to f on compact subsets of U.

Problem 6. [*10 points; 5, 5.*]

For $k \geq 1$, let $a_k = 1 - \frac{1}{k^2}$. For $n \geq 1$, define $f_n : \mathbb{D} \to \mathbb{D}$ by letting $f_n(z) = \prod_{k=1}^{n} \frac{a_k - z}{1 - a_k z}$.

(a) Prove that the sequence $\{f_n\}$ converges to an analytic function $f : \mathbb{D} \to \mathbb{D}$, uniformly on compact subsets of $\mathbb{D}$.

(b) Prove that there do not exist an open set $U \subset \mathbb{C}$ and an analytic function $g : U \to \mathbb{C}$ such that $\overline{\mathbb{D}} \subset U$, and $g(z) = f(z)$, for every $z \in \mathbb{D}$.

Problem 7. [*10 points.*]

Let $\gamma : [0.1] \to \mathbb{C}$ be a path such that $\gamma(0) = 1$ and $\gamma(t) \neq 0$, for every $t \in [0, 1]$. Assume that $(f_t, D_t)_{0 \leq t \leq 1}$ is an analytic continuation of $f_0(z) = \log z$ along γ. Prove that f_t is a branch of the logarithm, for every $t \in [0, 1]$.

Problem 8. [*10 points; 5, 5.*]

Let $u : \mathbb{C} \to \mathbb{R}$ be a harmonic function such that $\displaystyle\int\int |u(x+iy)|^2 \, \mathrm{d}x \, \mathrm{d}y < \infty$.

(a) Prove that $u(a) = \dfrac{1}{\pi r^2} \displaystyle\iint\limits_{B_r(a)} u(x+iy) \, \mathrm{d}x \, \mathrm{d}y$, for every $a \in \mathbb{C}$ and $r > 0$. Here, $B_r(a) = \{z \in \mathbb{C} \mid |z-a| < r\}$ denotes the open ball of radius r centered at a.

(b) Prove that $u(z) = 0$, for every $z \in \mathbb{C}$.