

Name:	_____
Student #:	_____
TA's Name:	_____
Session #:	_____

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**General guidelines:** You may cite without proof any theorem given in the text (unless explicitly stated otherwise). You need to reprove any result given in an exercise. If in doubt, please ask!

1. (15 points) Let  $\Omega$  be an open subset of  $\mathbb{C}$ . For compact subset  $K$  of  $\Omega$ , define the hull

$$\hat{K} = \{z \mid z \in \Omega, |f(z)| \leq \sup_{w \in K} |f(w)|, \text{ for every } f \in \mathcal{O}(\Omega)\}.$$

Let  $\hat{K}_c$  be the convex hull of  $K$ , namely the smallest convex subset of  $\mathbb{C}$  containing  $K$ . Show that

a)  $d(K, \mathbb{C} \setminus \Omega) = d(\hat{K}, \mathbb{C} \setminus \Omega)$ ;

b)  $\hat{K} \subset \hat{K}_c$ .

Hint: A set  $S \subset \mathbb{C}$  is convex if and only if  $tZ_1 + (1-t)Z_2 \in S$  for all  $t \in [0, 1]$ , provided that  $Z_1, Z_2 \in S$ .

2. (20 points) Suppose that  $u$  is a  $C^2$  subharmonic function on the whole complex plane.

a) Prove that for any positive  $R_1$  and  $R_2$  with  $R_2 > R_1$ ,

$$\int_0^{2\pi} (u(R_2 e^{i\theta}) - u(R_1 e^{i\theta})) d\theta \geq \int_{R_1 \leq |z| \leq R_2} \log \left( \frac{R_2}{|z|} \right) \Delta u \, dx dy.$$

b) Show that if  $u$  satisfies that

$$\limsup_{z \rightarrow \infty} \frac{u(z)}{\log |z|} = 0$$

then  $u$  must be a constant.

3. (15 points)  $f$  is an entire function. Assume that  $f(z + 1) = f(z)$  and  $|f(z)| \leq e^{C|z|}$  for some  $C < 2\pi$ . Show that  $f$  is a constant.

4. (15 points) Assume that  $f$  and  $g$  are entire functions of finite order  $\lambda$ . Assume that for a sequence  $a_n$ ,  $f(a_n) = g(a_n)$  and  $\sum |a_n|^{-(\lambda+\epsilon)} = \infty$  for some  $\epsilon > 0$ . Show that  $f = g$ .

5. (15 points) Suppose that a linear transformation carries one pair of concentric circles into another pair of concentric circles. Prove that the ratios of the radii must be the same.

6. (10 points) Let  $f$  be a holomorphic on the closed disc of radius  $R$  and assume that  $f(0) \neq 0$ . Let the zeros of  $f$  in the open disc be ordered by increasing absolute value,

$$z_1, z_2, \dots, z_N,$$

each zero being repeated according to its multiplicity. Prove that

$$|f(0)| \leq \frac{\sup_{|z|=R} |f(z)|}{R^N} |z_1 z_2 \cdots z_N|.$$

7. (10 points) For each  $\psi \in C_0^\infty(\mathbb{C})$  (the space of smooth functions with compact support) such that

$$\int \int_{\mathbb{C}} \psi(z) z^n dx dy = 0$$

there exists a  $u \in C_0^\infty(\mathbb{C})$  such that

$$\frac{\partial u}{\partial \bar{z}} = \psi.$$

END OF EXAM