Complex Analysis Qualifying Exam – Spring 2025

Name: ____________________________________

Student ID: ________________________________

Instructions:

No books or notes. You may use without proofs results proved in Conway, Chapters I-XI. However, if using a homework problem, please make sure you reprove it. Present your solutions clearly, with appropriate detail.

You have 180 minutes to complete the test.

Notation: $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
Total		60

Problem 1. [*10 points.*]

Let f be a holomorphic function in a neighborhood of the closed unit disc $\overline{\mathbb{D}}$, and suppose that

$$|f(0)| + |f'(0)| < \inf\{|f(z)| : |z| = 1\}.$$

Show that f has at least two zeros (counting multiplicity) in $\mathbb{D}$.

(*Hint:* Make use of the function $g(z) = f(0) + f'(0)z - f(z)$.)

Problem 2. [*10 points; 5, 5.*]

Let $\alpha \in (0, 1]$ and $\Omega_\alpha \subset \mathbb{C}$ denote the open region

$$\Omega_\alpha := \left\{ z = re^{i\theta} \colon \theta \in \left(-\frac{\alpha\pi}{2}, \frac{\alpha\pi}{2} \right),\ r > 0 \right\}.$$

(i) Determine for which α there is a Möbius transformation S from Ω_α onto $\mathbb{D}$. Prove nonexistence or give an explicit example if such exists.

(ii) Determine for which α there is a conformal map f from Ω_α onto $\mathbb{D}$. Prove nonexistence or give an explicit example if such exists.

Problem 3. [*10 points.*]

Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function in the unit disc. Assume f is injective and $f(0) = 0$. Prove that there exists a holomorphic function g in $\mathbb{D}$ such that $(g(z))^2 = f(z^2)$ for all $z \in \mathbb{D}$.

Problem 4. [*10 points.*]

Let $a \in \mathbb{D}$ and set $G = \mathbb{D} \setminus \{a\}$. Find all analytic automorphisms of G, i.e., find all one-to-one and onto analytic functions from G to G. Write down the expressions for such functions (can be unsimplified). Prove your answer.

Problem 5. [*10 points.*]

Let $f_n : \mathbb{D} \to \mathbb{C}$ be a bounded sequence of holomorphic functions, and let $\{z_m\}_{m\geq 1}$ be a sequence in $\mathbb{D}$ that converges in $\mathbb{D}$. Assume that $\lim_{n\to\infty} f_n(z_m)$ exists for all $m \geq 1$. Show that the sequence $\{f_n\}$ converges uniformly on compact subsets of $\mathbb{D}$.

Problem 6. [*10 points; 2, 2, 2, 4.*]

Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic and assume $f(z) \neq 0$ for $z \neq 0$. For $0 \leq r < 1$, let

$$M(r) = \max_{|z|=r} |f(z)|.$$

(i) Show that the function $h_s(z) = s \log |z| + \log |f(z)|$ is harmonic in $\mathbb{D} \setminus \{0\}$ for all $s \in \mathbb{R}$.

(ii) Show that there exists $s \in \mathbb{R}$ such that

$$\max_{|z|=\frac{1}{2}} h_s(z) = \max_{|z|=\frac{1}{8}} h_s(z).$$

(iii) Show that

$$M\left(\frac{1}{4}\right)^2 \leq M\left(\frac{1}{2}\right) M\left(\frac{1}{8}\right).$$

(iv) Show that equality holds if and only if $f(z) = az^n$ for some $a \in \mathbb{C}$ and some integer $n \geq 0$.