Complex Analysis Qualifying Exam – Spring 2026

Name: ______________________________

Student ID: ______________________________

Instructions:

No books or notes. You may use without proofs results proved in Conway, Chapters I-XI. However, if using a homework problem, please make sure you reprove it. Present your solutions clearly, with appropriate detail.

You have 180 minutes to complete the test.

Notation: $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.

Question	Score	Maximum
1		10
2		10
3		10
4		10
5		10
6		10
Total		60

Problem 1. [*10 points.*]

Let f be an entire function (i.e., analytic in the entire complex plane $\mathbb{C}$). Show that the series

$$g(z) := \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z), \qquad f^{(n)} := \frac{d^n f}{dz^n},$$

defines an entire function g.

Problem 2. [*10 points.*]

Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic such that

(i) $f(0) = 0$,

(ii) $|f(z)| < 2000$ for all $z \in \mathbb{D}$,

(iii) $f(z) = f(-z)$ for all $z \in \mathbb{D}$.

Prove that $\left|f\left(\frac{1}{45}\right)\right| < 1$.

Problem 3. [*10 points.*]

Prove that, for any $a \in \mathbb{C}$ and any integer $n \geq 2$, the polynomial $1 + z + az^n$ has at least one root in the disk $\{|z| \leq 2\}$.

Hint: The constant term in a monic polynomial is the product of its zeros (up to sign).

Problem 4. [*10 points.*]

Let f be holomorphic on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Assume that

(i) f extends continuously to $\overline{\mathbb{H}} \setminus \{0\}$,

(ii) $f(x) \in \mathbb{R}$ for all $x \in \mathbb{R} \setminus \{0\}$,

(iii) there exists an integer $N > 0$ such that $|f(z)| \leq |z|^N + \frac{1}{|z|^N}$ for all $z \in \mathbb{H}$.

Prove that there are constants $a_k \in \mathbb{R}$ such that $f(z) = \sum_{k=-N}^{N} a_k z^k$.

Problem 5. [*10 points.*]

Let $G = \{z \in \mathbb{C} : |z| > 1\}$. Let $u : \overline{G} \to \mathbb{R}$ be a continuous bounded function, such that u is harmonic in G. Assume

$$u(z) \leq 0 \text{ for all } |z| = 1.$$

Show that $u(z) \leq 0$ for all $z \in G$.

Problem 6. [*10 points; 5, 2, 3.*]

Let $A, B \subset \mathbb{C}$ be two disjoint nonempty compact sets. Consider the following property:

Property (P): For any two entire functions f, g, there exists a polynomial p such that

$$|p(z) - f(z)| < \frac{1}{1000} \text{ for } z \in A$$

and

$$|p(z) - g(z)| > 1000 \text{ for } z \in B.$$

(i) Assume $\mathbb{C} \setminus (A \cup B)$ is connected. Show that property (P) is true.

(ii) Show that property (P) holds for the compact sets

$$A = \{z \in \mathbb{C} : |z| = 1\}, \quad B = \{2\}.$$

In this case, $\mathbb{C} \setminus (A \cup B)$ is disconnected.

(iii) Exhibit two disjoint nonempty compact sets A, B for which property (P) is false.