Numerical Analysis Qualifying Exam
Spring 2021

May 13, 2021

Instructions:

- There are 8 problems, worth a total of 200 points.
- The 3 hour exam starts at
  - **Thursday, May 13 5:00 PM PST** (Friday, May 14 8:00 AM GMT+8)

  and concludes at
  - **Thursday, May 13 8:00 PM PST** (Friday, May 14 11:00 AM GMT+8).

  You then have 15 additional minutes to upload your work to Gradescope. After this, your work will be considered late and will **not be accepted**.
- You must work by yourself on these problems, with **no assistance** from other people. You are allowed all other resources that exist prior to this exam, including books and notes, online or otherwise, and calculators or computers, however, your answers should be written up in **your own words** and not copied from any source.
- Have pen or pencil ready, and enough paper. Also have ready the necessary equipment to promptly upload the finished pages onto Gradescope at the conclusion of the exam.
- Start each question on a separate page, upload clear and legible work, and properly label the page locations of each problem on Gradescope. If you do not, you risk losing points.
- You must show sufficient **detail** in your work to receive full credit.
- For questions, prior to or during the exam, email `lcheng@math.ucsd.edu`

1. (25 pts) Consider symmetric, positive definite $A \in \mathbf{R}^{n\times n}$, for $n \geq 2$, and let $A = LU$ denote its LU factorization. Furthermore, let $A^{(k)}$ denote the matrix appearing in Gaussian elimination on A prior to the kth step (so $A^{(1)} = A$ and $A^{(n)} = U$), and suppose we know $A^{(k)}(k:n, k:n)$ is symmetric, positive definite for all $1 \leq k \leq n$.

   Prove the following two properties about the entries of A and U:

   - $u_{ii} \leq a_{ii}$, for all $1 \leq i \leq n$;
   - $|u_{ij}| \leq \max_{1\leq k\leq n} a_{kk}$, for all $1 \leq i < j \leq n$.

2. (25 pts) Let nonsingular $A \in \mathbf{R}^{n\times n}$ satisfy $a_{ii} \neq 0$ for all $1 \leq i \leq n$, where $n \geq 2$. For $b \in \mathbf{R}^n$, consider the iterative method for solving $Ax = b$ with its sequence of approximations, $\{x^{(k)}\}_{k=1}^{\infty}$, generated by initial guess $x^{(0)} \in \mathbf{R}^n$ and the recursive formula

$$x_i^{(k+1)} = (1-\omega)x_i^{(k)} + \frac{\omega}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k+1)}\right),$$

   for $i = n, n-1, \ldots, 1$ and a choice of $\omega \in \mathbf{R}$ (note, this is not SOR).

   When $\omega \in (-\infty, 0] \cup [2, \infty)$, prove the iterative method does not converge, meaning: there exists an initial guess $x^{(0)}$ such that the sequence of approximations will not converge to the solution of $Ax = b$.

3. (25 pts) Suppose you have available for use an efficient algorithm that can obtain, for given $x \in \mathbf{R}^2$, an orthogonal matrix $B \in \mathbf{R}^{2\times 2}$ such that $e_1^T Bx = 0$, where $e_1^T = [1, 0]$, and suppose this algorithm uses M flops. Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \\ 0 & A_{32} \end{bmatrix} \in \mathbf{R}^{m\times n},$$

   where $A_{11} \in \mathbf{R}^{p\times p}$, $A_{22} \in \mathbf{R}^{(n-p)\times(n-p)}$, $2 \leq p \leq n-2$, and $m > n$. Consider using the available algorithm to efficiently calculate, by taking into account reduced operations involving 0's and 1's, the matrix $R \in \mathbf{R}^{m\times n}$, where R is upper triangular and satisfies the existence of orthogonal $Q \in \mathbf{R}^{m\times m}$ such that $A = QR$.

   Write down an (allowably unsimplified) expression for the exact number of flops needed, and extract and simplify its Cp^3 term, for constant $C \in \mathbf{R}$ independent of m, n, p.

4. (25 pts) Suppose you are given the two approximations $c_m < c_n$, for given $0 \le m < n$, coming from the bisection method applied to some continuous function using some starting interval (note, c_0 refers to the midpoint of the starting interval).

   Among all possible starting intervals that can satisfy this given information, find the minimum interval length. Be sure to prove your result is actually a minimum.

5. Fix an inner product for $C([a,b])$, where $a < b$. Now, for any $g \in C([a,b])$, define $p_g^{(k)}$ to be the degree $\le k$ polynomial that best approximates g in $[a,b]$ under the norm induced by the inner product. Consider the following statement:

   For all $f \in C([a,b])$ and $0 \le m < n$, then $p_f^{(m)} = p_q^{(m)}$, where $q = p_f^{(n)}$.

   Prove this statement or give a counterexample.

6. (25 pts) Let $f \in C^\infty([a,b])$, for some $a < b$, and let

$$a = x_0 < x_1 < \cdots < x_n = b$$

   be evenly spaced nodes with stepsize h. Suppose $f(x_i)$ and $f'(x_i)$ are given, for $0 \le i \le n$, and let $p(x)$ denote the piecewise cubic Hermite interpolating polynomial interpolating this data.

   Prove

$$\int_a^b f(x)\,dx - \int_a^b p(x)\,dx = Cf^{(j)}(\xi_n)h^k,$$

   for some $\xi_n \in [a,b]$, $C \in \mathbf{R}$ independent of h, and integers $j,k \ge 0$. Be sure to write out the C, j, k you get.

7. (25 pts) Consider the initial value problem with:

- ODE:
$$y' = f(t, y),$$
for $t \in [t_0, T]$ with $t_0 < T$, where f is continuous in
$$D = \{(t, y) \mid t \in [t_0, T], y \in (-\infty, \infty)\},$$
and Lipschitz continuous in variable y in D;
- Initial value $y(t_0) = y_0$.

Consider the predictor-corrector method of the form

$$y_{i+1} = \delta_0 y_{i-1} + \delta_1 y_i + h\phi_{f,h}(t_i, y_{i-1}, y_i),$$

using equal stepsize h and some $\delta_0, \delta_1 \in \mathbf{R}$, generated from:

- Predictor: a one-step explicit method with formula
$$y_{i+1} = y_i + h\psi_{f,h}(t_i, y_i),$$
satisfying:
  - $f \equiv 0 \Rightarrow \psi_{f,h} \equiv 0$, for all $h > 0$;
  - There exists $H > 0$ and K such that $h \in [0, H]$ implies
$$|\psi_{f,h}(t_i, z_i) - \psi_{f,h}(t_i, w_i)| \leq K|z_i - w_i|,$$
for all $t_i \in [t_0, T]$ and $z_i, w_i \in \mathbf{R}$.
- Corrector: linear implicit multistep method of the form
$$y_{i+1} = \frac{y_{i-1} + y_i}{2} + h(\gamma_0 f(t_{i-1}, y_{i-1}) + \gamma_2 f(t_{i+1}, y_{i+1})),$$
for some $\gamma_0, \gamma_2 \in \mathbf{R}, \gamma_2 \neq 0$.

Determine whether this method is zero-stable.

8. (25 pts) Consider the initial value problem with:

- ODE:
$$y' = f(t, y),$$
for $t \in [t_0, T]$ with $t_0 < T$, where f is continuous in
$$D = \{(t, y) \mid t \in [t_0, T], y \in (-\infty, \infty)\},$$
and Lipschitz continuous in variable y in D;
- Initial value $y(t_0) = y_0$.

Now fix $p \geq 1$ and suppose you are interested, in the case $y \in C^\infty([t_0, T])$, in developing an adaptive stepsize algorithm that makes use of approximations from the Taylor method of order p and from the Taylor method of order $p + 1$.

Write down in detail what a typical step of this adaptive stepsize algorithm looks like and explain its substeps. Be sure to include parts that are usually found in adaptive stepsize algorithms.