Numerical Analysis Qualifying Examination

September 8, 2006

NAME __________________

SIGNATURE __________________

#1	20	
#2	20	
#3	20	
Total	60	

Question 1. In this problem we will analyze the case of Lagrange interpolation on a set of distinct knots $x_0 < x_1 < \ldots < x_n$ with corresponding function values $f(x_i)$, $0 \le i \le n$.

a. Show how to construct the Lagrange interpolant $p_n(x)$ satisfying $p_n(x_i) = f(x_i)$, $0 \le i \le n$ using divided differences.

b. Prove

$$f(x) - p_n(x) = f[x, x_0, x_1, \ldots, x_n] \prod_{i=0}^{n} (x - x_i).$$

Question 2. Define the terms:

a. Consistency

b. Stability

c. Convergence

as they relate to a multistep formula. Apply these concepts to analyze the two step formula

$$y_{k+1} = y_{k-1} + 2hf(y_k)$$

Question 3. Let

$$\mathcal{I}(f) = \int_{-1}^{1} f(x) \approx w_1 f(x_1) + w_2 f(x_2) = \mathcal{Q}(f)$$

a. Compute the knots and weights such that $\mathcal{Q}(f)$ is the two point Gaussian quadrature formula.

b. Determine the order of the quadrature formula computed in part a.

c. Write down an expression for the error $|\mathcal{I}(f) - \mathcal{Q}(f)|$.

Numerical Analysis Qualifying Exam

Parts B and C

September 8, 2006

Name ______________________________

#1	20	
#2	20	
#3	20	
#4	20	
#5	20	
B-C	100	
A	60	
Total	160	

(20) 1. Let the *computed* L and U satisfy $A + E = LU$, where L is unit lower triangular and U is upper triangular. Derive the bound on E : $|E_{ij}| \leq (3+u)u \max(i-1, j)g$, $g = \max\limits_k \max\limits_{i,j} |a_{ij}^{(k)}|$.

(20) 2. Prove that $\hat{x}$ is a least squares solution to $r = Ax - b$, where A is $m \times n$ and $m \geq n$, iff $\hat{x}$ satisfies the normal equations.

(20) 3. (a) Prove that if A is positive semi-definite, then its eigenvalues are non-negative.

(b) Prove that if A is real symmetric, then A is positive definite iff its eigenvalues are positive.

(c) Let $B = \begin{bmatrix} A \\ a^T \end{bmatrix}$, where A is $m \times n$, $m \geqslant n$. Prove that $\sigma_n(B) \geqslant \sigma_n(A)$ and $\sigma_1(A) \leqslant \sigma_1(B) \leqslant \sqrt{\sigma_1(A)^2 + \|a\|_2{}^2}$.

(20) 4. Let A be $m \times n$.

(a) Prove that if $A^+ = \begin{cases} (A^TA)^{-1}A^T & \text{if } \operatorname{rank}(A) = n \\ A^T(AA^T)^{-1} & \text{if } \operatorname{rank}(A) = m. \end{cases}$

(b) Prove that $\|B(\lambda - A^+)\|_2 = \dfrac{\lambda}{\sigma_r(\sigma_r{}^2 + \lambda)}$, where $B(\lambda) = (A^TA + \lambda I)^{-1}A^T$, $\lambda > 0$, $m \geqslant n$, $\operatorname{rank}(A) = r$.

(20) 5. Let A be symmetric positive definite.

(a) Prove that $a_{ii} > 0$ for all i and $|a_{ij}| < (a_{ii} + a_{jj})/2$ for $i \neq j$.

(b) Prove that $A = LDL^T$ exists, where L is unit lower triangular and D is diagonal with positive diagonal elements.

(c) Prove that $\max\limits_k \max\limits_{i,j} |a_{ij}^{(k)}| = \max\limits_{i,j} |a_{ij}|$