Math 270abc: Numerical Analysis

Instructor: Randolph Bank

Fall Quarter 2010
Qualifying Examination
September 9, 2010

Name ______________________

Signature ______________________

#1	25	
#2	25	
#3	25	
#4	25	
#5	25	
#6	25	
#7	25	
#8	25	
Total	200	

Question 1. Let A be an $n \times n$ nonsingular matrix. We consider the solution of the linear system $Ax = b$. Suppose we have an approximate solution $\hat{x}$ to this system, and let $r = b - A\hat{x}$ be the residual. Prove

$$\frac{\|\hat{x} - x\|}{\|x\|} \leq Cond(A) \frac{\|r\|}{\|b\|}$$

where $\|\cdot\|$ is any vector norm, and $Cond(A)$ is the condition number of A with respect to the induced matrix norm.

Question 2. Let A be an $n \times n$ matrix. Prove (by induction) Schur's Theorem: $A = UTU^t$, where U is a unitary matrix, and T is upper triangular.

Question 3. Let A and B be symmetric, positive definite $n \times n$ matrices. Assume there exist positive constants a and b such that

$$a \leq \frac{x^t A x}{x^t B x} \leq b$$

for all $x \neq 0$. Consider the solution of $Ax = c$ by the iterative method:

$$B(x_{k+1} - x_k) = \omega(c - Ax_k)$$

where x_0 is given and $\omega = 2/(a+b)$. Prove:

$$\|e_k\|_A \leq \left(\frac{b-a}{b+a}\right)^k \|e_0\|_A$$

where $e_k = x - x_k$, and $\|x\|_A^2 = x^t A x$.

Question 4. Let $\phi(x)$ be a scalar function of the vector variable x. Suppose $\phi(x)$ is continuous with continuous first and second partial derivatives, and suppose that the Hessian is symmetric and uniformly positive definite.

a. Formally define Newton's method with line search for solving the optimization problem $\min_x \phi(x)$.

b. Let p_k be the Newton search direction. Show that $\partial\phi(x_k + \alpha p_k)/\partial\alpha < 0$ at $\alpha = 0$. Why is this fact significant for the line search?

Question 5. Let

$$\mathcal{I}(f) = \int_{-1}^{1} f(x)dx$$

We consider a Gaussian quadrature formula of the form

$$\mathcal{Q}(f) = w_1 f(x_1) + w_2 f(x_2)$$

a. Compute the weights w_i and knots x_i to maximize the order.

b. Using the Peano Kernel Theorem, prove

$$|\mathcal{I}(f) - \mathcal{Q}(f)| \leq C_0 \|f^{iv}\|_{\infty[-1,1]}.$$

c. Derive a composite formula $\mathcal{Q}_c(f)$, using the basic rule $\mathcal{Q}$, for approximating

$$\mathcal{I}(f) = \int_{a}^{b} f(x)dx$$

on a uniform mesh of n intervals with $h = (b-a)/n$,

d. Prove

$$|\mathcal{I}_c(f) - \mathcal{Q}_c(f)| \leq C_0(b-a)\left(\frac{h}{2}\right)^4 \|f^{iv}\|_{\infty[a,b]}$$

Question 6. In this problem we will analyze the case of continuous piecewise *quadratic* interpolation on a mesh of $n+1$ knots $x_0 < x_1 < \cdots < x_n$. We will also need the interval midpoints $x_{i+1/2} = (x_i + x_{i+1})/2$.

a. Show the the dimension of the space $\mathcal{S}$ of continuous piecewise quadratic polynomials is $N = 2n+1$.

b. We will use the *nodal* basis functions. There are two types: *hat functions*, which satisfy

$$\phi_i(x_j) = \delta_{ij} \qquad \phi_i(x_{j+1/2}) = 0 \qquad 0 \le i \le n,$$

and *bump functions*, which satisfy

$$\phi_{i+1/2}(x_j) = 0 \qquad \phi_{i+1/2}(x_{j+1/2}) = \delta_{ij} \qquad 0 \le i \le n-1.$$

Draw a picture of both types of basis functions.

c. Let f^* be the continuous piecewise quadratic interpolant for f. Prove, using the Peano Kernel Theorem,

$$\|f - f^*\|_2 \le Ch^3 \|f'''\|_2$$
$$\|f' - f^{*\prime}\|_2 \le Ch^2 \|f'''\|_2$$

Question 7. Let $y' = f(y)$, $y(0) = y_0$, $x_k = kh$, $k = 0, 1, \ldots$, where $h > 0$ is fixed. Let $p \geq 0$, $r \geq 0$ be integers. From

$$\int_{x_{k-p}}^{x_{k+1}} y'\, dx = \int_{x_{k-p}}^{x_{k+1}} f(y)\, dx$$

we get

$$y(x_{k+1}) = y(x_{k-p}) + \int_{x_{k-p}}^{x_{k+1}} f(y)\, dx$$

A multistep method is obtained by interpolating f at $x_{k+1}, x_k, \ldots, x_{k-r+1}$ by a polynomial of degree r, and then integrating that interpolating polynomial exactly.

a. Prove that any multistep method derived in this fashion is *consistent*.

b. Prove the scheme is *stable* is $p = 0$ and *weakly stable* is $p > 0$.

Question 8. Prove Gronwall's Lemma: Let

$$y' \leq \kappa y + \tau$$

for $0 \leq t \leq T$, and $\tau, \kappa, y \geq 0$, τ and κ constant. Then

$$\max_{0 \leq t \leq T} y(t) \leq e^{\kappa T} y(0) + \frac{\tau}{\kappa} \left(e^{\kappa T} - 1 \right).$$