Ph.D./Masters Qualifying Examination
in Numerical Analysis

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9am–12pm
Friday May 26, 2017

Name ____________________

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- Put your name in the box provided and staple exam to your solutions.
- Write your name clearly on every sheet of paper you submit.

1 Numerical Linear Algebra (270A)

Question 1.1. Let $A \in \mathbb{R}^{n\times n}$, and let $\|\cdot\|_p$ denote the standard l^p norms on $\mathbb{R}^n$, $1 \le p \le \infty$.

(a) Show the following norm equivalence relations for the l^p-norms on $\mathbb{R}^n$:

$$\|u\|_\infty \le \|u\|_2 \le \|u\|_1 \le \sqrt{n}\|u\|_2 \le n\|u\|_\infty, \quad \forall u \in \mathbb{R}^n,$$

and then show the following induced matrix norm and spectral radius relationships:

$$\|A\|_1 \le \sqrt{n}\|A\|_2 \le n\|A\|_1, \qquad \|A\|_\infty \le \sqrt{n}\|A\|_2 \le n\|A\|_\infty, \qquad \rho(A) \le \|A\|_p.$$

(b) Assume A is invertible and use the results from (a) to derive analogous relationships for $\kappa_p(A)$.

(c) Assume A is invertible, and $Ax = b$ and $A(x + \delta x) = (b + \delta b)$ for some $x, b, \delta x, \delta b \in \mathbb{R}^n$. Show

$$\frac{\|\delta x\|_p}{\|x\|_p} \le \kappa_p(A)\frac{\|\delta b\|_p}{\|b\|_p}, \qquad \frac{\|\delta b\|_p}{\|b\|_p} \le \kappa_p(A)\frac{\|\delta x\|_p}{\|x\|_p}, \qquad 1 \le p \le \infty.$$

Question 1.2. Let $A \in \mathbb{R}^{m\times n}$, $m \ge n$, and consider the overdetermined system:

$$Ax = b, \quad \text{where } x \in \mathbb{R}^n, b \in \mathbb{R}^m.$$

(a) Formulate the minimization problem that defines the least-squares solution, and derive the normal equations from this problem.

(b) Show that $A^T A$ is nonsingular if and only if A has full rank.

(c) Identify the projector P arising in least-squares, and show how to exploit a QR factorization.

Question 1.3. Let $A, B \in \mathbb{R}^{n\times n}$ be SPD matrices.

(a) Show that A defines an inner-product and norm

$$(u, v)_A = (Au, v)_2, \qquad \|u\|_A = (u, u)_A^{1/2},$$

where $(u, v)_2$ is the usual Euclidean 2-inner-product.

(b) Starting with the Caley-Hamilton Theorem, derive the Conjugate Gradient method for solving the preconditioned linear system: $BAu = Bf$. Mathematically justify each step of the derivation. The derivation will be based around building up an expanding set of Krylov subspaces, exploiting a 3-term recursion for generating an A-orthogonal bases for these subspaces, and enforcing minimization of the A-norm of the error at each iteration of the method.

2 Numerical Approximation and Nonlinear Equations (270B)

Question 2.1. Let $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable on an open convex set D.

(a) Derive the following expansion with integral remainder:

$$F(x+h) = F(x) + F'(x)h + \int_0^1 \left\{ F'(x+\xi h) - F'(x) \right\} h \, d\xi,$$

and then use this expansion to derive Newton's method for $F(x) = 0$.

(b) Assume that $F(x^*) = 0$ for some $x^* \in D$, and that $F'(x^*)$ is nonsingular. Prove the basic convergence theorem for Newton's method: There exists an open neighborhood $S \subset D$ containing x^* such that, for any $x_0 \in S$, the Newton iterates are well-defined, remain in S, and converge to x^* at q-superlinear rate.

(c) Show that if the Jacobian $F'(x)$ is Lipschitz in the set S in part (b) for some uniform Lipschitz constant, then the convergence rate is q-quadratic.

Question 2.2. Let $f \colon \mathbb{R}^n \to \mathbb{R}$, $c \colon \mathbb{R}^n \to \mathbb{R}^m$, $0 < m < n$, and consider the problem:

$$\min_{x \in \mathbb{R}^n} f(x),$$
$$\text{subject to } c(x) = 0.$$

Using some basic ideas from linear algebra, prove the main result that leads to the method of Lagrange Multipliers for this problem: If f and c are differentiable at a feasible point x^*, then

$$\nabla f(x^*)^T p \geq 0, \quad \forall p \text{ such that } c'(x^*)p = 0,$$

if and only if there exists a vector $\lambda^* \in \mathbb{R}^m$ such that $\nabla f(x^*) = c'(x^*)^T \lambda^*$.

Question 2.3. Consider the following tabulated information about a function $f \colon \mathbb{R} \to \mathbb{R}$:

x	f(x)
0	1
1	1
2	9

(a) Construct the (unique) quadratic interpolation polynomial $p_2(x)$ which interpolates the data.

(b) If the function $f(x)$ that generated the above data was actually the cubic polynomial $P_3(x) = 2x^3 - 2x^2 + 1$, derive an error bound (a fairly "tight" one) for the interval $[0, 2]$.

(c) Use the composite trapezoid rule with two intervals to construct an approximation to:

$$\int_0^2 f(x) \, dx,$$

and give an expression for the error.

3 Numerical Ordinary Differential Equations (270C)

Question 3.1. We consider now the problem of best L^p-approximation of a function $u(x) = x^4$ over the interval $[0, 1]$ from a subspace $V \subset L^p([0, 1])$.

(a) Determine the best L^2-approximation in the subspace of linear functions; i.e., $V = \text{span}\{1, x\}$, and justify the technique you use.

(b) Precisely formulate the best approximation problem in the case $p \neq 2$, and propose an algorithm for finding the solution.

(c) Let X be a general Hilbert space, and let $U \subset X$ be a subspace. Prove that the orthogonal projection of u onto $Pu \in U$ is the best approximation, and that this projection is unique.

Question 3.2. Consider the initial value problem in ordinary differential equations:

$$\begin{aligned} y' &= f(t, y), \quad t \in (a, b) \\ y(a) &= \alpha. \end{aligned}$$

(a) Derive the Taylor method of order 2 using Taylor expansion of the solution to the ODE.

(b) Derive the Runge-Kutta method of order 2 (by matching terms in the Taylor method).

(c) Consider the multistep method (3-step Adams-Bashford):

$$\begin{aligned} w_0 &= \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \\ w_{i+1} &= w_i + \frac{h}{12}[23f(t_i, w_i) - 16f(t_{i-1}, w_{i-1}) + 5f(t_{i-2}, w_{i-2})], \quad i = 2, 3, \ldots, N-1. \end{aligned}$$

Determine the local truncation error, and examine the stability using the root condition. Finally, draw a conclusion about the convergence properties of the method.