Ph.D./Masters Qualifying Examination in Numerical Analysis

Examiner: Michael Holst

9am–12pm
Wednesday September 13, 2017

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#1.1	25	
#1.2	25	
#1.3	25	
#2.1	25	
#2.2	25	
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Total	200	

- Put your name in the box provided and staple exam to your solutions.
- Write your name clearly on every sheet of paper you submit.

1 Numerical Linear Algebra (270A)

Question 1.1. Let $A \in \mathbb{R}^{n\times n}$, and let $\|\cdot\|_p$ denote the standard l^p norms on $\mathbb{R}^n$, $1 \le p \le \infty$. We know that the following norm equivalence relations for the l^p-norms on $\mathbb{R}^n$ can be shown to hold:

$$\|u\|_\infty \le \|u\|_2 \le \|u\|_1 \le \sqrt{n}\|u\|_2 \le n\|u\|_\infty, \quad \forall u \in \mathbb{R}^n.$$

(a) Show the following induced matrix norm and spectral radius relationships:

$$\|A\|_1 \le \sqrt{n}\|A\|_2 \le n\|A\|_1, \qquad \|A\|_\infty \le \sqrt{n}\|A\|_2 \le n\|A\|_\infty, \qquad \rho(A) \le \|A\|_p.$$

(b) Give a precise mathematical definition of a *well-posed* problem, and a precise mathematical definition of the *condition* of a problem.

(c) Assume A is invertible, and $Ax = b$ and $A(x + \delta x) = (b + \delta b)$ for some $x, b, \delta x, \delta b \in \mathbb{R}^n$. Show

$$\frac{\|\delta x\|_p}{\|x\|_p} \le \kappa_p(A)\frac{\|\delta b\|_p}{\|b\|_p}, \qquad \frac{\|\delta b\|_p}{\|b\|_p} \le \kappa_p(A)\frac{\|\delta x\|_p}{\|x\|_p}, \qquad 1 \le p \le \infty.$$

Question 1.2. Let $A \in \mathbb{R}^{m\times n}$, $m \ge n$, and consider the overdetermined system:

$$Ax = b, \qquad \text{where } x \in \mathbb{R}^n, b \in \mathbb{R}^m.$$

(a) Formulate the minimization problem that defines the least-squares solution, and rigorously derive the normal equations from this problem.

(b) Assume A has full rank. Identify the projector P arising in least-squares, show it is idempotent, and show how to exploit a QR factorization of A in an algorithm for finding the least-squares solution.

(c) If $P \in \mathbb{R}^{m\times m}$ is nonzero and idempotent, show that $\|P\|_2 \ge 1$, and that equality holds when P self-adjoint.

Question 1.3. Let $A, B \in \mathbb{R}^{n\times n}$ be SPD matrices.

(a) Show that A defines an inner-product and norm

$$(u, v)_A = (Au, v)_2, \qquad \|u\|_A = (u, u)_A^{1/2},$$

where $(u, v)_2$ is the usual Euclidean 2-inner-product. Now show that BA and $E = I - BA$ are A-self-adjoint, that they have real eigenvalues, and further that BA is A-positive.

(b) Derive the basic linear method (BLM) for solving $Au = f$, starting with any $u^0 \in \mathbb{R}^n$:

$$u^{k+1} = (I - BA)u^k + Bf, \quad k = 0, 1, 2, \ldots$$

(c) Prove the basic convergence theorem for the BLM: *If A and B are SPD, then*

$$\rho(I - \alpha BA) = \|I - \alpha BA\|_A < 1$$

if and only if $\alpha \in (0, 2/\rho(BA))$. Moreover, convergence is optimal when $\alpha = 2/[\lambda_{min}(BA) + \lambda_{max}(BA)]$, giving

$$\rho(I - \alpha BA) = \|I - \alpha BA\|_A = 1 - \frac{2}{1 + \kappa_A(BA)}.$$

2 Numerical Approximation and Nonlinear Equations (270B)

Question 2.1. Let $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable on an open convex set D.

(a) Rigorously derive the following expansion with integral remainder:

$$F(x+h) = F(x) + F'(x)h + \int_0^1 \left\{F'(x+\xi h) - F'(x)\right\} h \, d\xi,$$

and then use this expansion to derive Newton's method for $F(x) = 0$.

(b) Give a complete algorithm (in pseudocode only) for implementing Newton's method for the solving problem: $F(x) = 0$. Include backtracking line-search (i.e., damping) and allow for inexact solves of the linearized systems at each step.

(c) Assume that $F(x^*) = 0$ for some $x^* \in D$, and that $F'(x^*)$ is nonsingular. Prove the basic convergence theorem for Newton's method: There exists an open neighborhood $S \subset D$ containing x^* such that, for any $x_0 \in S$, the Newton iterates are well-defined, remain in S, and converge to x^* at q-superlinear rate.

Question 2.2. Let $f\colon \mathbb{R}^n \to \mathbb{R}$, $c\colon \mathbb{R}^n \to \mathbb{R}^m$, $0 < m < n$, and consider the problem:

$$\min_{x \in \mathbb{R}^n} f(x),$$
$$\text{subject to } c(x) = 0.$$

(a) Give the first-order necessary condition for constrained optimality, and clearly specify nonlinear system of equations that must be solved to find a point satisfying the first order necessary condition.

(b) Derive the jacobian matrix of the nonlinear system of equations from part (a), and use this jacobian to write down a complete Newton's method algorithm for solving the nonlinear system you specified in part (a).

(c) Give the second-order necessary and sufficient condition for constrained optimality.

Question 2.3. Consider the following tabulated data for a function $f : \mathbb{R} \to \mathbb{R}$:

x	f(x)
0	1
1	3
2	13

(a) Construct the (unique) quadratic interpolation polynomial $p_2(x)$ which interpolates the data.

(b) If the function $f(x)$ that generated the above data was actually the cubic polynomial $P_3(x) = x^3 + x^2 + 1$, derive an error bound for the interval $[0, 2]$.

(c) Use the composite trapezoid rule with two intervals to construct an approximation to:

$$\int_0^2 f(x) \, dx,$$

and give an expression for the error.

3 Numerical Ordinary Differential Equations (270C)

Question 3.1. We turn to best and near-best approximation in Banach and Hilbert spaces.

(a) Consider first the problem of best L^p-approximation of a function $u(x) = x^2 - 2x^3$ over the interval $[0,1]$ from a subspace $V \subset L^p([0,1])$. Determine the best L^2-approximation in the subspace of linear functions; i.e., $V = \text{span}\{1, x\}$. Give an outline of an algorithm that could find the best L^p approximation when $p \neq 2$.

(b) Let X be a general Hilbert space, and let $U \subset X$ be a subspace. Prove that the orthogonal projection of u onto $Qu \in U$ is the unique best approximation of u in U, i.e., that Qu uniquely satisfies

$$\|u - Qu\|_X = \inf_{w \in U} \|u - w\|_X.$$

Rather than the orthogonal projection from part (b), consider now the "Galerkin projection" of u onto $\bar{u} = Pu \in U$ as defined by the problem:

$$\text{Find } \bar{u} \in U \subset X \text{ such that } A(\bar{u}, \bar{v}) = F(\bar{v}), \quad \forall \bar{v} \in U \subset X,$$

where $A(u, v)$ is a bounded and coercive bilinear form on the Hilbert space X, and $F(v)$ is an element of the dual space X^* to X.

(c) Prove that $\bar{u}$ exists and is unique, and that Cea's Lemma holds for $\bar{u}$:

$$\|u - \bar{u}\|_X \leq C \inf_{w \in U} \|u - w\|_X.$$

(I.e., this shows that $\bar{u}$ is a "quasi-best" approximation to u.)

Question 3.2. Consider the following initial value problem (IVP):

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha. \tag{3.1}$$

(a) Assume for this part only that $f(t, y) = t^3 y - 2$, $a = 0$, $b = 1$, $\alpha = 1$. Now, rigorously prove that this problem is well-posed.

Consider now the following class of one-step methods ($\theta \in [0,1]$) for (3.1):

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + h[\theta f(t_i, w_i) + (1 - \theta) f(t_{i+1}, w_{i+1})] \end{aligned}$$

(b) Determine truncation error for this class of methods.

(c) For problem (3.1), what ranges of θ make the method consistent, stable, unstable, and/or conditionally stable? What are the region of stability for the cases $\theta = 0$ and $\theta = 1$?