Numerical Analysis Qualifying Examination

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May 21, 2004

NAME ____________________

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#2	25	
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Total	150	

Question 1. Let A be an $n \times n$ symmetric positive definite matrix. Consider the partitioning

$$A = \begin{pmatrix} \alpha & c^t \\ c & B \end{pmatrix}$$

where α is a scalar and B is $n-1 \times n-1$.

a. Prove the (Schur complement) matrix $B - cc^t/\alpha$ is positive definite.

b. Using part a, prove by induction that $A = LDL^t$, where L is unit lower triangular, and D is diagonal with positive diagonal elements.

Question 2. Let $x \in \mathbb{R}^n$, and $x \neq 0$.

a. Find a Householder transformation Q such that $Qx = \sigma e_1$ where σ is a scalar and e_1 the unit vector $e_1^T = (1\,0\,0 \ldots 0)$.

b. Show how to obtain an orthogonal matrix Q such that $Qx = \sigma e_1$ using a sequence of Givens rotations.

Question 3. Let A and B be symmetric, positive definite $N \times N$ matrices. Assume there exist positive constants α and β such that

$$\alpha \leq \frac{x^t A x}{x^t B x} \leq \beta$$

for all $x \neq 0$. Consider the solution of $Ax = b$ by the iterative method:

$$B(x_{k+1} - x_k) = \omega(b - Ax_k)$$

where x_0 is given and $\omega = 2/(\alpha + \beta)$.

a. Derive the error propagator G for this iteration.

b. Prove:

$$\|e_k\|_A \leq \left(\frac{\beta - \alpha}{\beta + \alpha}\right)^k \|e_0\|_A$$

where $e_k = x - x_k$.

Question 4. Let $F(x)$ be a vector function of a vector variable x. Here we study the solution of $F(x^*) = 0$ by Newton's method. Assume that $F(x)$ is continuously differentiable with Jacobian matrix $J(x) \equiv \partial F/\partial x$. Assume that $J(x)$ has a uniformly bounded inverse ($\|J^{-1}\| \leq \gamma$), and is Lipschitz continuous ($\|J(x) - J(y)\| \leq L\|x - y\|$).

a. Define Newton's method for solving $F(x^*) = 0$.

p. Let $e_k = x^* - x^k$ denote the error. Prove

$$\|e_{k+1}\| \leq \frac{\gamma L}{2}\|e_k\|^2.$$

Hint: you may assume the identity

$$F(y) = F(x) + \int_0^1 J(x + \theta(y - x))(y - x)\, d\theta$$

Question 5. Let $f \in C^2(I)$, $I = [a, b]$, and let $x_i = a + ih$, $0 \le i \le n$, $h = (b-a)/n$ be a uniform mesh on I. Let $\mathcal{S}$ be the space of continuous piecewise linear polynomials with respect to this uniform mesh and let $\tilde{f}$ denote the continuous piecewise linear polynomial interpolant of f.

a. Compute the dimension of $\mathcal{S}$ and define the standard *nodal basis* functions $\{\phi_i\}$ for $\mathcal{S}$.

b. Using the Peano Kernel Theorem, prove:

$$\|f - \tilde{f}\|_{\mathcal{L}^2(I)} \le Ch^2 \|f''\|_{\mathcal{L}^2(I)}$$

(You do NOT need to explicitly evaluate the constant C.)

Question 6. Let $y' = f(y), y(0) = y_0$. Assume $|f(w) - f(z)| \le \mathcal{K}|w - z|$ for all $w, z \in \mathcal{R}$, and the stepsize h is constant. Let $x_k = kh$, and $y_k \approx y(x_k)$, $k = 0, 1, \dots$ be the approximate solution generated by the Predictor-Corrector scheme based on Euler's method and the Backward Difference method

$$y_k^* = y_{k-1} + hf(y_{k-1})$$
$$y_k = y_{k-1} + hf(y_k)$$

a. Show how y_k and y_k^* can be combined to estimate the local truncation error (L.T.E.) ℓ_k for the Backward Difference method.

b. Find the region of absolute stability for each method. Note whether each method is A-stable and/or L-stable.

c. Using a (discrete) Gronwall lemma, prove that, for h sufficiently small,

$$\max_k |y(x_k) - y_k| \le Ch\|y''\|_\infty$$