Ph.D./Masters Qualifying Examination in Numerical Analysis

Examiners: Philip E. Gill and Bo Li

1–4pm
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6402 AP&M

NAME ____________________

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- Add your name in the box provided and staple this page to your solutions.
- Write your name clearly on every sheet submitted.

1. Norms, Condition Numbers and Linear Equations

Question 1.1.

(a) Let $\mathbf{u}$ denote the unit roundoff, and assume that $n\mathbf{u} < 1$ for the positive integer n. If $\{\delta_i\}$ are n scalars such that $|\delta_i| \le \mathbf{u}$, prove that

$$\prod_{i=1}^{n}(1+\delta_i) = 1 + \theta_n, \quad \text{where} \quad |\theta_n| \le \gamma_n,$$

with $\gamma_n = n\mathbf{u}/(1 - n\mathbf{u})$.

(b) State the *standard rounding-error model* for floating-point arithmetic. Given representable numbers a and b, compute the backward and forward relative error for the floating-point value $\hat{s}$ of the expression $s = \sqrt{a^2 + b^2}$. (You may assume that the square root function conforms to the standard rounding-error model for floating-point arithmetic.)

Question 1.2.

(a) Define the spectral condition number $\mathrm{cond}_2(A)$ for any $A \in \mathbb{R}^{m\times n}$. State (but do not prove) an expression for $\mathrm{cond}_2(A)$ in terms of the singular values of A.

(b) Let $A \in \mathbb{R}^{n\times n}$ be nonsingular. Find a solution of the problem

$$\min_{E\in\mathbb{R}^{n\times n}} \{\|E\|_2 : A + E \;\; \text{singular}\}.$$

(c) Prove that $1/\mathrm{cond}_2(A)$ is the relative two-norm distance of A to the nearest singular matrix.

Question 1.3. Assume that A is an $m \times n$ matrix with rank k ($k < \min(m, n)$).

(a) Define what is meant by a *full-rank factorization* $A = BC$.

(b) State the full-rank factorization of A in terms of the QR decomposition with column interchanges. (You may assume that the decomposition is computed in exact arithmetic.)

(c) Using the QR decomposition of part **(b)**, define bases for the subspaces range(A) and null(A). Prove that the proposed bases satisfy the properties of a subspace basis.

(d) Using the QR decomposition of part **(b)**, define orthogonal projections onto range(A) and null(A). Prove that the proposed projections satisfy the properties of an orthogonal projection.

(e) Derive the pseudoinverse of A in terms of the full-rank factorization of part **(b)**.

2. Nonlinear Equations and Optimization

Question 2.1. Let A denote an $n \times n$ matrix, and let s and y be arbitrary n-vectors.

(a) Find all the eigenvalues of the matrix $I + \gamma uv^T$, where γ is a scalar and u and v are n vectors.

(b) Consider the Broyden update formula

$$A_+ = A + \frac{1}{s^T s}(y - As)s^T.$$

If $\|\cdot\|_F$ denotes the Frobenius norm, show that A_+ minimizes $\min \|B - A\|_F$ over all B such that $Bs = y$.

(c) If A is nonsingular, find a condition on A, s and y that will ensure that A_+ is nonsingular.

Question 2.2. Let $f : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}$ be twice differentiable on an open convex set $\mathcal{D}_0 \subseteq \mathcal{D}$.

(a) State the first- and second-order *necessary* conditions for $x^* \in \mathbb{R}^n$ to be an unconstrained minimizer of f.

(b) State first and second-order *sufficient* conditions for $x^* \in \mathbb{R}^n$ to be an unconstrained minimizer of f.

(c) Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ such that

$$f(x) = (x_3 - 1)^2 \sin x_1 + x_1^2 + x_2^2 - \pi x_1.$$

(i) Write down the quadratic model $q(x)$ that interpolates f, ∇f and $\nabla^2 f$ at the point $x_0 = (-\pi/2,\, 0,\, \pi + 1)^T$.

(ii) Find the step p_N from x_0 to a stationary point of the quadratic model.

(iii) Determine if p_N is a descent direction for f at x_0.

(iv) Find a descent direction of negative curvature at x_0 (if one exists).

Question 2.3. Given an $n \times n$ symmetric matrix B and vectors y and s, consider the symmetric rank-one formula

$$B_+ = B + \frac{1}{(y - Bs)^T s}(y - Bs)(y - Bs)^T.$$

(a) Let $f(x)$ be a quadratic function with positive-definite Hessian H. Let $s = x_+ - x$ and $y = \nabla f(x_+) - \nabla f(x)$. If vectors $\bar{s} = \bar{x}_+ - \bar{x}$ and $\bar{y} = \nabla f(\bar{x}_+) - \nabla f(\bar{x})$ satisfy $B\bar{s} = \bar{y}$, show that $B_+\bar{s} = \bar{y}$.

(b) Show that if B is symmetric and positive definite, then B_+ will be positive definite if and only if

$$\frac{y^T B^{-1} y - y^T s}{y^T s - s^T B s} > 0.$$

3. Approximation and Numerical ODEs

In this part, we assume that $a, b \in \mathbb{R}$ with $a < b$. We also denote by $\mathcal{P}_n$ the set of all polynomials of degree $\leq n$ for any integer $n \geq 0$.

Question 3.1.

(a) Let $n \geq 0$ be an integer and T_n the nth Chebyshev polynomial of first kind. Let $P \in \mathcal{P}_n$ satisfy that $|P(x)| \leq 1$ for all $x \in [-1, 1]$. Show that

$$|P(y)| \leq |T_n(y)| \qquad \forall y \notin [-1, 1].$$

(b) Let $\mathbb{F}$ denote the class of functions $a_0 + a_1 \cos x + a_2 \cos 2x$ with $a_0,\ a_1,\ a_2 \in \mathbb{R}$. Find $T \in \mathbb{F}$ such that

$$\int_0^\pi |T(x) - x|^2 dx \leq \int_0^\pi |S(x) - x|^2 dx \qquad \forall S \in \mathbb{F}.$$

Question 3.2.

(a) Use the error formula for the Lagrange interpolation of $f \in C^2[a, b]$ at the two points a and b to derive the error for the trapezoidal numerical integration rule

$$\int_a^b f(x)\, dx \approx \frac{1}{2}(b-a)\left[f(a) + f(b)\right].$$

(b) Assume that $f \in C^2[a, b]$. Derive an error formula for the composite trapezoidal numerical integration rule

$$\int_a^b f(x)\, dx \approx \frac{h}{2}\left[f(a) + f(b)\right] + h \sum_{j=1}^{N-1} f(x_j).$$

Here $N \geq 1$ is an integer, $h = (b-a)/N$, and $x_j = a + jh$ $(j = 0, \ldots, N)$.

(c) Apply the composite trapezoidal numerical integration rule to

$$\int_0^{10} \sin x\, dx.$$

How large N is needed so that the error of the numerical integration is less than 10^{-6}? (Ignore the round-off error.) Justify your answer.

3. Approximation and Numerical ODEs

In this part, we assume that $a, b \in \mathbb{R}$ with $a < b$. We denote by $\mathcal{P}$ the set of all real polynomials. For any integer $n \geq 0$, we denote by $\mathcal{P}_n$ the set of all real polynomials of degree $\leq n$.

Question 3.1.

(a) Let $f \in C[a, b] \setminus \mathcal{P}$. For any integer $n \geq 0$, denote

$$E_n(f) = \min_{p_n \in \mathcal{P}_n} \max_{a \leq x \leq b} |f(x) - p_n(x)|.$$

Prove that the sequence $\{E_n(f)\}_{n=0}^{\infty}$ is *strictly* decreasing (i.e., $E_n(f) > E_{n+1}(f)$ for all $n \geq 0$) and converges to 0.

(b) Find the real numbers A, B, and C so that the numerical quadrature

$$\int_{-2}^{2} f(x)\, dx \approx Af(-1) + Bf(0) + Cf(1)$$

has the highest possible degree of precision. What is this highest possible degree of precision?

Question 3.2. Let $x_1^{(n)}, \ldots, x_n^{(n)}$ be the n distinct roots of orthogonal polynomials Q_n in $L^2(a, b)$ $(n = 1, 2, \ldots)$.

(a) For each $n \geq 2$, let $l_1^{(n)}, \ldots, l_n^{(n)}$ be the Lagrange basis polynomials associated with $x_1^{(n)}, \ldots, x_n^{(n)}$. Prove the following:

$$\int_a^b l_j^{(n)}(x) l_k^{(n)}(x)\, dx = 0 \qquad \text{if } 1 \leq j, k \leq n, \text{ and } j \neq k;$$

$$\sum_{k=1}^{n} \int_a^b \left[l_k^{(n)}(x)\right]^2 dx = b - a.$$

(b) Let $L_{n-1} : C[a, b] \to \mathcal{P}_{n-1}$ be the Lagrange interpolation operator associated with $x_1^{(n)}, \ldots, x_n^{(n)}$. Prove that

$$\lim_{n \to \infty} \int_a^b \left[f(x) - (L_{n-1}f)(x)\right]^2 dx = 0 \qquad \forall f \in C[a, b].$$