Numerical Analysis Qualifying Examination

Instructors: Ioana Dumitriu, Melvin Leok

Spring 2025

NAME: ______________________________

Part I: Math 270A

Problem	**Points Possible**	**Points Earned**
1	25	
2	25	
3	25	
Total	75	

1. Let A be an $n \times n$ complex matrix,

$$A = \begin{bmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}.$$

Consider the matrix B that results from rotating A $180°$ around its center:

$$B = \begin{bmatrix} a_{nn} & \cdots & \cdots & a_{n1} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{1n} & \cdots & \cdots & a_{11} \end{bmatrix}.$$

(a) (15 points) Show that if A is positive definite, B is positive definite.

(b) (10 points) Assume now that n is large and A has no special properties. Describe the most efficient algorithm for deciding if A is positive definite, and state its asymptotic complexity as a function of n. Is this algorithm stable?

2. Let A be an $m \times n$ real matrix and B be a $m \times m$ real matrix.

(a) (15 points) Find, with proof, a $n \times m$ matrix X which minimizes $||AX - B||_F$, where $|| \cdot ||_F$ is the Frobenius norm.

(b) (10 points) State and explain any additional properties your minimizer X may have, if any.

3. Let A be a real $n \times n$ matrix with positive eigenvalues, for which $\lambda_1(A) = 1$, where $\lambda_1(A)$ denotes the largest eigenvalue of A.

Consider the iterative method given by

$$((\mu+1)I_n - \mu A)x_k + \mu b = (\mu+1)x_{k+1} \ , \qquad x_0 \in \mathbb{R}^n \ .$$

(a) (15 points) Prove that, regardless of x_0, the method converges to $\bar{x}$, the solution to $A\bar{x} = b$, for any $0 < \mu < \infty$.

(b) (10 points) Assume now that $0 < \mu < \infty$. Explain why, in general, this method is expected to beat direct methods for solving $Ax = b$ from a complexity standpoint.

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Part II: Math 270BC

Problem	Points Possible	Points Earned
4	25	
5	25	
6	25	
7	25	
8	25	
Total	125	

4. The divided difference of order $k > 1$ of $f(x)$ is defined by

$$f[a_1, \dots, a_k] = \frac{f[a_2, \dots, a_k] - f[a_1, \dots, a_{k-1}]}{a_k - a_1}, \qquad \text{where } f[a_1] = f(a_1).$$

(a) (12 points) Prove that

$$f[a_1, \dots, a_k] = \sum_{i=1}^{k} \frac{f(a_i)}{(a_i - a_1) \cdots (a_i - a_{i-1})(a_i - a_{i+1}) \cdots (a_i - a_k)},$$

and thus deduce that the divided difference is invariant under permutation of the arguments.

(b) (13 points) Show that

$$f[a_1, \dots, a_{k-1}, x] = f[a_1, \dots, a_k] + (x - a_k) f[a_1, \dots, a_k, x]$$

and use this result to derive the formula

$$\begin{aligned} f(x) = & f[a_1] + (x - a_1) f[a_1, a_2] + (x - a_1)(x - a_2) f[a_1, a_2, a_3] \\ & + \cdots + (x - a_1)(x - a_2) \cdots (x - a_{n-1}) f[a_1, \dots, a_n] + E_n(x), \end{aligned}$$

where $E_n(x) = p_n(x) f[a_1, \dots, a_n, x]$.

5. (25 points) Prove that if $a_i = k^{-1}(t_{i+1} + t_{i+2} + \cdots + t_{i+k})$, then

$$\sum_{i=-\infty}^{\infty} a_i B_i^k(x) = x, \qquad k \geq 1.$$

Recall that $B_i^k = V_i^k B_i^{k-1} + (1 - V_{i+1}^k) B_{i+1}^{k-1}$, where $V_i^k(x) = \frac{x - t_i}{t_{i+k} - t_i}$, and

$$B_i^0(x) = \begin{cases} 1 & \text{if } t_i \leq x < t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

6. (25 points) Let $\langle f, g\rangle = \int_a^b w(x)f(x)g(x)dx$, where $w(x) \geq 0$ is a given weight function on $[a, b]$. Prove that the sequence of polynomials defined below is orthogonal with respect to the inner product $\langle \cdot\,, \cdot\rangle$,

$$p_n(x) = (x - a_n)p_{n-1}(x) - b_n p_{n-2}(x), \qquad n \geq 1,$$

where

$$\begin{aligned} p_0(x) &= 1, \\ p_1(x) &= x - a_1, \\ a_n &= \langle xp_{n-1}, p_{n-1}\rangle / \langle p_{n-1}, p_{n-1}\rangle, \\ b_n &= \langle xp_{n-1}, p_{n-2}\rangle / \langle p_{n-2}, p_{n-2}\rangle. \end{aligned}$$

7. A s-step method has the form

$$\sum_{m=0}^{s} a_m y_{n+m} = h \sum_{m=0}^{s} b_m f(t_{n+m}, y_{n+m}),$$

where $\rho(w) \equiv \sum_{m=0}^{s} a_m w^m$ and $\sigma(w) \equiv \sum_{m=0}^{s} b_m w^m$. Consider a s-step method with $\sigma(w) = \beta w^{s-1}(w+1)$ and order s.

(a) (15 points) Using the formula,

$$\rho(w) - \sigma(w)\ln(w) = \mathcal{O}(|w-1|^{s+1}), \quad w \to 1,$$

explicitly derive the methods for $s = 2$ and $s = 3$.

(b) (10 points) Are the two methods you derived convergent? Justify your answer.

8. Consider the following collocation Runge–Kutta method:

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \hline & \frac{1}{4} & \frac{3}{4} \end{array}$$

(a) (10 points) Compute the stability function $R(h\lambda)$ for the collocation Runge–Kutta method you derived in the part (a), which satisfies the equation

$$y_{n+1} = R(h\lambda) y_n$$

when the Runge–Kutta method is applied to the model problem,

$$\begin{cases} y'(t) = \lambda y(t), & t > 0, \\ y(0) = 1. \end{cases}$$

(b) (8 points) Is the method A-stable (the domain of absolute stability contains $\mathbb{C}^-$)? Justify your answer.
Hint: You may use the following Lemma: *Let R be an arbitrary rational function that is not a constant. Then $|R(z)| < 1$ for all $z \in \mathbb{C}^-$ if and only if all the poles of R have positive real parts and $|R(it)| \leq 1$ for all $t \in \mathbb{R}$.*

(c) (7 points) Determine the order of accuracy of the collocation Runge–Kutta method. (You may use the Theorem relating the order of collocation methods and the orthogonality properties of the polynomial with the collocation points as its roots)