1. Analysis Qualifying Exam, Fall 2004

Instructions: Clearly explain and justify your answers. You may cite theorems from the text, notes, or class as long as they are not what the problem explicitly asks you to prove. You may also use the results of prior problems or prior parts of the same problem when solving a problem – this is allowed even if you were unable to prove the previous results. Make sure to state the results that you are using and be sure to verify their hypotheses. All problems have **equal** value. You are to do 7 out of the 8 problems. Please clearly indicate on the front of your exam which 7 if the 8 problems you want me to grade.

Notation: Let m denote Lebesgue measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ where $\mathcal{B}_{\mathbb{R}}$ is the Borel σ – algebra on $\mathbb{R}$. We will write dx for $dm(x)$, (f, g) for $\int_{\mathbb{R}} f(x)\, g(x)\, dx$ and $\mathcal{B}_{[-1,1]}$ for the Borel σ – algebra on $[-1, 1]$.

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1. Find, with justification, the values of following limits and integrals:

(a)
$$\lim_{n\to\infty} \int_0^\infty \cos\left(\frac{x}{n}\right) e^{-x} dx$$

(b)
$$\int_0^\infty \left[\int_0^\infty \frac{x}{1+x^2} e^{-xt} dx\right] dt$$

(c)
$$\lim_{n\to\infty} \int_{-\infty}^\infty e^{-|x+n|} dx.$$

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2. Let $g \in L^1(\mathbb{R}, m)$ be chosen so that $\int_{\mathbb{R}} g(x)\, dx = 3$. Find, with justification, the following limit;

$$\lim_{n\to\infty} \int_{\mathbb{R}} g(x) \sin^2(nx)\, dx.$$

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3. Let f and g be two real $L^2(\mathbb{R}, m)$ – functions. Show

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x)\, g(x-n)\, dx = 0. \tag{1.1}$$

Hint: First prove Eq. (1.1) holds if $g \in L^2(\mathbb{R}, m)$ is further assumed to have compact support. Under this assumption, notice that the following identity holds for sufficiently large M,

$$\begin{aligned}\int_{\mathbb{R}} f(x)\, g(x-n)\, dx &= \int_{\mathbb{R}} \left[f(x)\, 1_{[-M,M]}(x-n)\right] \cdot g(x-n)\, dx \\ &= \left(f(\cdot)\, 1_{[-M,M]}(\cdot - n), g(\cdot - n)\right).\end{aligned}$$

4. Suppose $T_n : X \to Y$ for $n \in \mathbb{N}$ is a sequence of bounded linear operators between two Banach spaces, X and Y. Further assume that $\lim_{n\to\infty} T_n x$ exists for all $x \in X$. Show $Tx := \lim_{n\to\infty} T_n x$ defines a bounded linear operator from X to Y.

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5. Suppose that $\{\nu_n\}_{n=1}^{\infty}$ are complex measures and μ is a finite positive measure on a measurable space, $(X, \mathcal{M})$. Further let $|\nu_n|$ denotes the total variation measure associated to ν_n.

(a) If $\sum_{n=1}^{\infty} |\nu_n|(X) < \infty$, then $\nu := \sum_{n=1}^{\infty} \nu_n$ is a complex measure.

(b) If $|\nu_n(A)| \le \mu(A)$ and $\nu(A) := \lim_{n\to\infty} \nu_n(A)$ exists for all $A \in \mathcal{M}$, then $\nu : \mathcal{M} \to \mathbb{C}$ is a complex measure.

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6. Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of absolutely continuous functions such that $f_n' \in L^1(\mathbb{R}, m)$ and $c := \lim_{n\to\infty} f_n(0)$ exists in $\mathbb{R}$. Further assume there exists $g \in L^1(\mathbb{R}, m)$ such that $\lim_{n\to\infty} \int_{\mathbb{R}} |g(x) - f_n'(x)| dx = 0$.

(a) Show that $f(x) := \lim_{n\to\infty} f_n(x)$ exists for all $x \in \mathbb{R}$.

(b) Show that f is absolutely continuous and $f'(x) = g(x)$ for m- a.e. x.

Hint: As usual, integration can be used to prove theorems about differentiation.

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7. Let $h \in C([-1,1], \mathbb{R})$ be a one to one function and let Z denote the space of functions of the form $p(h) := \sum_{k=1}^{n} a_k h^k$ for some $n \in \mathbb{N}$ and $a_k \in \mathbb{R}$. Show Z is a dense subspace of $L^1([-1,1], \mathcal{B}_{[-1,1]}, m)$. **Note:** if necessary, for partial credit, you may further assume that h is never zero.

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8. Let $G : \mathbb{R} \to \mathbb{R}$ be a bounded Borel measurable function. Define $f_0(t) = 1$ and $f_n : [-1,1] \to \mathbb{R}$ inductively by

$$f_{n+1}(t) = 1 + \int_0^t G(f_n(\tau))\, d\tau.$$

Show:

(a) f_n are well defined and that $f_n \in C([-1,1], \mathbb{R})$ for all n.

(b) The sequence $\{f_n\}_{n=1}^{\infty}$ has a uniformly convergent subsequence.