MATH 240 Qualifying Exam

May 14, 2024

Instructions: 3 hours, open book/notes (only Folland or personal lecture notes; no HW or other solutions). You may use without proofs results proved in Folland Chapters 1-8. Present your solutions clearly, with appropriate detail.

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1. (25 pts) Let $\{r_n\}_{n=1}^{\infty}$ be a sequence with $r_n \in [0,1]$ and define the function

$$f(x) := \sum_{r_n < x} \frac{1}{2^n}.$$

Show that f is Borel measurable, find all its points of discontinuity, and find $\int_0^1 f(x)dx$.

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2. (40 pts) (a) Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f_n, f \in L^1(\mu)$ $(n \in \mathbb{N})$ be non-negative functions. If $f_n \to f$ almost everywhere and $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$, show that $f_n \to f$ in $L^1(\mu)$.

(b) Does the conclusion in (a) continue to hold when we drop the hypothesis that f_n, f are non-negative? Either prove this or find a counter-example.

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3. (40 pts) Let $0 < \alpha \leq 1$ and let $\Lambda_\alpha([0,1])$ denote the space of Hölder continuous functions of exponent α on $[0,1]$. Specifically, $\Lambda_\alpha([0,1]) = \{f \in C([0,1]) : \|f\|_{\Lambda_\alpha} < \infty\}$ where

$$\|f\|_{\Lambda_\alpha} = |f(0)| + \sup_{x \neq y \in [0,1]} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

(a) Show that $\|\cdot\|_{\Lambda_\alpha}$ is a norm on $\Lambda_\alpha([0,1])$ and that with this norm $\Lambda_\alpha([0,1])$ is a Banach space.

(b) Let $B = \{f \in \Lambda_\alpha([0,1]) : \|f\|_{\Lambda_\alpha} \leq 1\}$ be the closed unit ball in $\Lambda_\alpha([0,1])$. Show that B is compact with respect to the uniform norm.

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4. (35 pts) Let A be a set and $1 < p < \infty$. Prove that a sequence $f_n \in \ell^p(A)$ converges weakly to $f \in \ell^p(A)$ if and only if (f_n) converges to f pointwise and $\sup_n \|f_n\|_p < \infty$.

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5. (25 pts) Let X be a set equipped with the discrete topology, and let $X^* = X \cup \{\infty\}$ be the one-point compactification of X. Let μ be a Radon measure on X^* and define the support of μ as

$$\text{supp}(\mu) = \bigcap \{N : N \subseteq X^* \text{ is closed and } \mu(N^c) = 0\}.$$

Prove that $\text{supp}(\mu)$ is countable.

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6. (35 pts) Let $f \in L^2(\mathbb{R}^n)$ be such that $f(x) = 0$ for a.e. $x \in \mathbb{R}^n \setminus A$, where $m(A) < \infty$. Show that, for any measurable $E \subset \mathbb{R}^n$,

$$\int_E |\hat{f}(\xi)|^2 d\xi \leq m(A)m(E)\|f\|_{L^2}^2.$$