QUALIFYING EXAM: REAL ANALYSIS

Thursday, May 15, 2025 (180 minutes).

Please turn all cell phones off completely and put them away.
No books, notes, or electronic devices are permitted during this exam.
You must show your work to receive credit.
Present your solutions clearly, and indicate which result you are using at every step.
You may use without proof any result (theorem, proposition, lemma, corollary) contained in Chapters 1-6 and 8 of *Gerald B. Folland, Real Analysis*.

Name (print): ____

Student ID number: ____

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1. [15 pts] Let $f \in L^1([0,1])$. Prove that the following are equivalent:

(1) $f \in L^2([0,1])$,

(2) there exists g absolutely continuous on $[0,1]$ such that for every $x, y \in [0,1]$ it holds

$$\left| \int_x^y f(t)\, dt \right|^2 \le (g(y) - g(x))\,(y - x)\,.$$

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2. [15 pts] Let $\mathcal{B}_{\mathbb{R}^n}$ be the Borel σ-algebra on $\mathbb{R}^n$ and let $\mu\colon \mathcal{B}_{\mathbb{R}^n} \to [0,\infty]$ be a measure such that

(1) $\mu(\mathbb{R}^n) = 1$,
(2) $\mu(\{x\}) = 0$ for every $x \in \mathbb{R}^n$.

Prove that for every $\epsilon > 0$ there is $\delta > 0$ such that

$$\operatorname{diam}(E) = \sup\{|x-y| \,:\, x, y \in E\} < \delta \qquad \Rightarrow \qquad \mu(E) < \epsilon\,.$$

(Hint: Start with $E \subset \overline{B_K(0)}$...)

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3. [20 pts] Let E be the Banach space $C([0,1])$ endowed with the uniform norm, $\|\cdot\|_u$. Let $(f_h)_{h\in\mathbb{N}} \subset E$ be a sequence and $f \in E$.

(1) Prove that if f_h converges weakly to f then there is $M > 0$ such that $|f_h(x)| \leq M$ for every $x \in [0,1]$ and $h \in \mathbb{N}$.
(2) Prove that if f_h converges weakly to f then $f_h(x) \to f(x)$ for every $x \in [0,1]$.
(3) $f_h(x) = x^h$ doesn't converge weakly in E.
(4) E is not reflexive.

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4. [15 pts] Let E be the Banach space $L^1([0,1])$ endowed with the L^1-norm, $\|\cdot\|_1$, and

$$C := \left\{ u \in E \,:\, u(x) \geq 0 \text{ a.e. } x \in [0,1]\,,\ \int_0^1 x\, u(x)\, dx \geq 1 \right\}.$$

Show that

(1) C is nonempty, closed and convex in E.
(2) $d(0,C) := \inf\{\|u\| \,:\, u \in C\} = 1$ (Hint: try with piecewise constant functions...)
(3) There is no $u \in C$ such that $\|u\| = d(0,C) = 1$.

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5. [20 pts] Let $f \in C^1(\mathbb{R})$ such that $f(0) = f(1)$ and $\displaystyle\int_0^1 f(x)\,\mathrm{d}x = 0$.

(1) Prove that $\displaystyle 4\pi^2 \int_0^1 |f(x)|^2\,\mathrm{d}x \leq \int_0^1 |f'(x)|^2\,\mathrm{d}x$.

(2) Prove that if $\displaystyle 4\pi^2 \int_0^1 |f(x)|^2\,\mathrm{d}x = \int_0^1 |f'(x)|^2\,\mathrm{d}x$, then there exist $a, b \in \mathbb{C}$ such that $f(x) = a\cos(2\pi x) + b\sin(2\pi x)$, for every $x \in [0, 1]$.

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6. [15 pts] Let $(\xi_m)_{m\geq 1}$ be a sequence of vectors in a Hilbert space $\mathcal{H}$. Assume that $\|\xi_m\| \leq 1$ and $\lim_{n\to\infty} \langle \xi_n, \xi_m \rangle = 0$, for every $m \geq 1$. Prove that $\lim_{n\to\infty} \langle \xi_n, \xi \rangle = 0$, for every $\xi \in \mathcal{H}$.

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