Math 281 Qualifying Exam. September 9, 2002

(1) Let a sequence of random variables be given by $X_n = 1/n$. Also,

let $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$. Show that $X_n \xrightarrow{L} 0$ but $f(X_n) \overset{L}{\not\rightarrow} f(0)$.

(2) State a version of the Slutsky theorem under which it is true that.

$$X_n \xrightarrow{L} 0 \Rightarrow f(X_n) \xrightarrow{L} f(0).$$

In this connection, what goes wrong in Problem (1) ?

(3) Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. random variables with sample mean and sample variance given by $\overline{X}_n$ and S_n^2, respectively.

a. If the underlying distribution is $N(\mu, \sigma^2)$, with both parameters unknown, what is the distribution of

$$\frac{\sqrt{n-1}\,(\overline{X}_n - \mu)}{S_n} \quad ?$$

Develop confidence intervals for μ in this case.

b. Now suppose we have a random sample from *any* distribution with finite mean and variance. In this case, derive the limiting distribution of

$$\frac{\sqrt{n-1}\,(\overline{X}_n - \mu)}{S_n} \quad .$$

Explain how the large sample confidence intervals for μ differ from those obtained in part a.

c. What is the limiting distribution of

$$\left(\frac{\sqrt{n-1}\,(\overline{X}_n - \mu)}{S_n}\right)^2 \quad ?$$

(Be certain to qualify your answer.)

d. Derive large sample confidence intervals for the variance σ^2.

Question (4) X is data from a statistical problem $\mathcal{F}$.

a. What does it mean for a statistic $T(X)$ to be complete and sufficient for $\mathcal{F}$?

b. State the factorization criterion for sufficiency.

c. Let $X_1, \dots, X_n$ $(n \geq 3) \sim_{iid} f \in \mathcal{F}$, the class of all densities on $\mathbb{R}$. If $\pi = P[X_1 \geq -1]$ find the UMVUE δ of π^3.

d. (continuing c.) Consider the subclass $\mathcal{F}_0 \subset \mathcal{F}$ consisting of uniform distributions $\{U[\theta-1, \theta+1]\}_{\theta \in \mathbb{R}}$. If it is known that $f \in \mathcal{F}_0$ explain whether δ above is still UMVU for this problem.

Question (5). $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ and $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ are independent.
$\mathrm{corr}(X_1, X_2) = \mathrm{corr}(Y_1, Y_2) = \rho$. Show that $\mathrm{corr}(X_1+X_2; Y_1+Y_2) = \rho$.
Could the same conclusion hold if, instead of independence, all four random variables were positively correlated?

Question (6). Consider estimating ξ^2 from iid. data $X_i \sim \mathcal{N}(\xi, 1)$; $i = 1, \dots, n$. Estimator $T_1 = \bar{X}^2$

a. What does Jensen's inequality tell you about the bias of T_1?

b. Explain whether T_1 is the MLE for ξ^2?

c. In what sense is the bias of T_1 "removable"?

d. Construct an unbiased estimator T_2 as an observable function of T_1 and explain whether T_2 is (i) UMVU ; (ii) admissible or not.

e. Show that if $\xi \neq 0$ T_1 and T_2 are asymptotically equivalent in the sense of MSE risk. (If you wish assume the formula $\mathrm{var}\, \bar{X}^2 = \frac{2}{n^2} + 4\frac{\xi}{n}$, although there is small extra credit for deriving it.)