Qualifying Exam: Mathematical Statistics 281AB

Spring 2025

Instructions: Provide rigorous and detailed mathematical arguments. Each statement should be supported by formal definitions, theorems, and proofs.

Total time allowed: **3 hours**. Total points: **50**.

Monotone Likelihood Ratio (MLR). Let $\{f(x;\theta) : \theta \in \Theta \subset \mathbb{R}\}$ be a family of probability densities (or mass functions) and let $T(X)$ be a statistic. The family is said to have the *MLR property in* $T(X)$ if, for every pair $\theta_1 < \theta_2$,

$$\Lambda_{\theta_2,\theta_1}(x) \;:=\; \frac{f(x;\theta_2)}{f(x;\theta_1)} \quad \text{is non-decreasing in } T(x).$$

Interpretation: larger values of $T(X)$ provide increasingly stronger evidence in favour of larger parameter values.

One–parameter exponential family. A distribution belongs to the *(natural) exponential family* if its density (or pmf) can be written

$$f(x;\theta) \;=\; h(x)\,\exp\bigl\{\eta(\theta)\,T(x) - A(\theta)\bigr\}, \qquad \theta \in \Theta,$$

where $h(x)$ — base measure (does not involve θ); $T(x)$ — canonical (sufficient) statistic; $\eta(\theta)$ — natural parameter (often required to be strictly monotone for MLR results) and $A(\theta)$ — log-partition function ensuring the density integrates to 1.

Problems

1. **General Properties of MLR Families** [20 pts]

   (i) Formally prove or disprove: If a family has the MLR property in a statistic $T(x)$, then $T(x)$ is necessarily a sufficient statistic. [10 pts]

   (ii) Consider the standard Cauchy distribution with location parameter θ:

   $$f(x;\theta) = \frac{1}{\pi(1+(x-\theta)^2)}, \quad x, \theta \in \mathbb{R}$$

      (i) Provide a rigorous argument demonstrating that the Cauchy family cannot be represented in exponential-family form (no finite-dimensional natural sufficient statistic exists). [5 pts]

      (ii) Prove rigorously that the standard Cauchy distribution does not possess the MLR property in any real-valued statistic $T(x)$ (Show that for *every* statistic $T(X)$ the MLR property fails). [5 pts]

2. **Asymptotics for Weibull Distribution** [30 pts]

   Fix $\xi < 0$. Consider the one–parameter family of densities

   $$f(x;\xi) \;=\; \big(1+\xi x\big)^{-1-\frac{1}{\xi}} \, \exp\!\big\{-(1+\xi x)^{-1/\xi}\big\}, \qquad x < -\frac{1}{\xi}, \;\; \xi < 0. \tag{1}$$

   This is the *Weibull* (finite–upper–endpoint) sub-family of the generalised extreme–value (GEV) distributions with fixed location $\mu = 0$ and scale $\sigma = 1$.

   Let $X_1, \dots, X_n \overset{\text{i.i.d.}}{\sim} f(\,\cdot\,;\xi_0)$ with an unknown true shape parameter $\xi_0 < 0$. Denote by

   $$\hat{\xi}_n \;=\; \arg\max_{\xi<0} \sum_{i=1}^{n} \log f(X_i;\xi)$$

   the *maximum-likelihood estimator* (MLE) of ξ_0.

   (i) **Consistency of the MLE.** [10 pts]
   Prove that $\hat{\xi}_n \to \xi_0$ in probability (you may invoke the general Z-estimation theorem, but all steps—identifiability, stochastic equicontinuity, etc.—must be verified explicitly for the density (1)).

   (ii) **Failure of the $\sqrt{n}$ rate.** [10 pts]
   Show that

   $$\sqrt{n}\,(\hat{\xi}_n - \xi_0) \;\xrightarrow{P}\; \infty,$$

   i.e. any $\sqrt{n}$-normal limit would be degenerate. (Hint: bound $\hat{\xi}_n - \xi_0$ below by a function of $X_{(n)}$ and control the tail of $X_{(n)}$.)

   (iii) **Correct scaling and non-Normal limit.** [10 pts]
   Define

   $$Z_n = n|\xi_0|\,\big(\hat{\xi}_n - \xi_0\big).$$

   Show converges in distribution to a non-Gaussian random variable Z. Prove that

   $$Z \;\overset{d}{=}\; -\log E, \qquad \text{where } E \sim \text{Exp}(1).$$

(That is, Z has the standard *Gumbel* distribution, so the MLE's limit law is *not Normal.*)

Hints:

- Consider data transformation $U_i = \big(1+\xi_0 X_i\big)^{-1/\xi_0}$, and its order statistics.
- Consider expanding the *local log-likelihood contrast*

$$\Delta_n(z) = \ell_n\Big(\xi_0 + \tfrac{z}{n|\xi_0|}\Big) - \ell_n(\xi_0), \qquad \ell_n(\xi) = \sum_{i=1}^{n} \log f(X_i;\xi).$$

  for the purpose of Asymptotic distribution.
- Then rewrite it as

$$\Delta_n(z) = \underbrace{\big[\log f\big(X_{(n)};\xi_0+h_n\big) - \log f\big(X_{(n)};\xi_0\big)\big]}_{\Delta_n^{\max}(z)} + \underbrace{\sum_{i=1}^{n-1}\big[\log f\big(X_{(i)};\xi_0+h_n\big) - \log f\big(X_{(i)};\xi_0\big)\big]}_{\Delta_n^{\text{bulk}}(z)}.$$

- Consider showing the decomposition

$$\boxed{\Delta_n^{\max}(z) = z \;-\; T_n e^{-z} + r_n(z)}, \qquad T_n = ne^{-U_{(n)}},\; U_{(n)} := (1+\xi_0 X_{(n)})^{-1/\xi_0},$$

  for $h_n = z/(n|\xi_0|) = O(n^{-1})$ and with a remainder $r_n(z) \xrightarrow{p} 0$ for each *fixed* $z < T_n$. If $z \geq T_n$ the likelihood is $-\infty$.
- Consider Let $h_n = z/(n|\xi_0|)$, $z \in \mathbb{R}$ fixed, and then show

$$\boxed{\Delta_n^{\text{bulk}}(z) = O_p\Big(\tfrac{z^2}{n}\Big).}$$

  by establishing

$$\Delta_n^{\text{bulk}}(z) = h_n S_{n-1}^{(1)} + \tfrac{1}{2}h_n^2 S_{n-1}^{(2)} + \sum_{i=1}^{n-1} R_n(X_{(i)}).$$

  for $g(x;\xi) = \log f(x;\xi)$ the partial derivatives $g_k(x) = \partial_\xi^k g(x;\xi_0)$ and $|R_n(x)| \leq \frac{1}{6}\sup_{\xi\in[\xi_0,\xi_0+h_n]}|g_3(x;\xi)|\,|h_n|^3$. Continue to show

$$h_n S_{n-1}^{(1)} = \frac{z}{n|\xi_0|}\, O_p(\sqrt{n}) \;=\; O_p\big(n^{-1/2}\big).$$

$$\tfrac{1}{2}h_n^2 S_{n-1}^{(2)} = \frac{z^2}{2n^2|\xi_0|^2}\,\big[\mathbb{E}g_2(X_1)\,n + O_p(\sqrt{n})\big] = O\big(z^2/n\big).$$

- Use this fact $\sup_{|\xi-\xi_0|\leq\varepsilon}|g_3(x;\xi)| < C(1+|x|^\alpha)$ for some $\alpha < \infty$,
- Consider For $x < -\frac{1}{\xi}$ and $\xi < 0$ write

$$\log f(x;\xi) = -\Big(1+\frac{1}{\xi}\Big)\underbrace{\log\big(1+\xi x\big)}_{=:L(x,\xi)} \;-\; \underbrace{\big(1+\xi x\big)^{-1/\xi}}_{=:E(x,\xi)}.$$

  and consider Taylor's expansions of the L and E terms separately.

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Let $X_1, \ldots, X_n$ be a random sample from the exponential distribution with location parameter θ and scale parameter λ. That is, the density function of Exponential(λ, θ) is

$$f_{\lambda,\theta}(x) = \frac{1}{\lambda} e^{-(x-\theta)/\lambda} \mathbf{1}\{x \geq \theta\}, \ \lambda > 0, \ \theta \in \mathbb{R}.$$

where $\mathbf{1}\{\cdot\}$ is the indicator function. These facts may be helpful in solving the problems.

Fact (1): $X_{(1)} = \min\{X_1, \ldots, X_n\} \sim$ Exponential$(\lambda/n, \theta)$.

Fact (2): $T = \sum_{i=1}^{n}(X_i - \theta) \sim$ Gamma(n, λ) or $(2T)/\lambda \sim \chi^2_{2n}$. The density of T is

$$f_T(t) = \frac{1}{\lambda^n \Gamma(n)} t^{n-1} e^{-t/\lambda} \mathbf{1}\{t \geq 0\}.$$

Fact (3): $\sum_{i=1}^{n}(X_i - X_{(1)}) \sim$ Gamma$(n-1, \lambda)$. $\sum_{i=1}^{n}(X_i - X_{(1)})$ and $X_{(1)}$ are independent.

For each of the following problems, you should either compute the boundary values of the critical region or justify how they can be determined.

(a) Suppose the location parameter θ is known. Derive the UMPU test of size α for testing $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda \neq \lambda_0$. Is the UMPU test you derived also a UMP test in this setting?

(b) Suppose the scale parameter λ is known. Derive the UMP test of size α for testing $H_0 : \theta \geq \theta_0$ versus $H_1 : \theta < \theta_0$. Calculate the probability of Type II error of this test when $\theta = \theta_0/2$.

(c) Suppose both θ and λ are unknown. Derive the UMPU test of size α for testing $H_0 : \lambda \leq \lambda_0$ versus $H_1 : \lambda > \lambda_0$.

(d) Suppose both θ and λ are unknown. Derive the UMPU test of size α for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$.