

## 2004 Fall Topology Qual

Two and a half hour exam. Each question is worth five marks. You can use any standard facts or theorems in your work provided you state them clearly.

1. Find a space  $X$  that has the same integral homology and fundamental group as the torus  $S^1 \times S^1$ , but is not homotopy equivalent to the torus. Prove that  $X$  is not homotopy equivalent to the torus.
2. Consider the standard covering projection  $S^n \rightarrow \mathbb{R}P^n$  which maps antipodal points to the same point in  $\mathbb{R}P^n$ . Prove that the covering projection is not null homotopic.
3. Show that  $\mathbb{R}P^k$  is not a retract of  $\mathbb{R}P^n$  for  $k < n$ .
4. Let  $M$  be a compact connected nonorientable 3-manifold. Show the first integral homology group of  $M$  is infinite.
5. Prove the Borsuk-Ulam theorem that if  $n > m \geq 1$ , then there is no map  $g : S^n \rightarrow S^m$  which commutes with the antipodal map.
6. Let  $M^4$  be a closed connected simply-connected 4-manifold. Show that  $H_1(M; \mathbb{Z}) = H_3(M; \mathbb{Z}) = 0$  and that  $H_2(M; \mathbb{Z})$  is a free abelian group.
7. Describe the universal cover of  $X = \mathbb{R}P^3 \vee S^2$ , and use it to compute the abelian group  $\pi_2(X)$ .
8. Let  $X$  be the space obtained by identifying the edges of a solid hexagon as shown below. Compute  $H_*(X; \mathbb{Z})$ .

