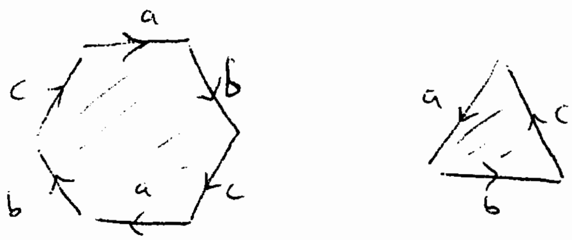
2009 Fall Topology Qual

Three hour exam. Each question is worth five marks. You can use any standard facts or theorems in your work provided you state clearly that you are doing so. Please try to write good clear mathematics!

1. Show that the fundamental group, based at the identity, of a topological group G is abelian.

2. A solid hexagon and a solid triangle are glued together along their edges, according to the following scheme. Calculate the fundamental group and the homology of the resulting space X.



3. Let Y be a space obtained by attaching a 4-ball, via a degree 6 map of its boundary, to a 3-sphere. Calculate the integral homology $H_*(Y \times \mathbb{R}P^2; \mathbb{Z})$.

4. The Euler characteristic $\chi(X)$ of a space X is defined as the alternating sum of the dimensions of the rational homology groups $H_i(X; \mathbb{Q})$. Use Poincaré duality to show that the Euler characteristic of a compact connected closed orientable 3-manifold M^3 is zero. Prove that the result still holds even if M is non-orientable.

5. Show that any homotopy equivalence from $\mathbb{C}P^{2n}$ to itself is orientation-preserving, that is has degree $+1$.

6. Calculate the homotopy group $\pi_3(\mathbb{R}P^4 \vee S^3)$ (where $\vee$ denotes the one-point union of the two spaces).

7. Let M^3 be a *homology sphere*: a connected closed compact 3-manifold with the same homology groups as S^3. Calculate the fundamental group and homology of the suspension ΣM? Use this to show that the suspension is homotopy-equivalent to S^4.

8. Let F_n denote the free group on n generators. Use covering space theory to prove that F_2 contains subgroups isomorphic to F_n, for every $n \geq 1$.

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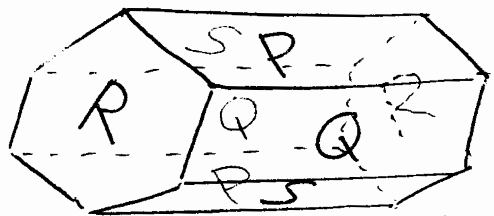
1. Construct a space whose integral homology groups are $\mathbb{Z}, \mathbb{Z}_5, \mathbb{Z}_5, \mathbb{Z}$ in dimensions $0, 1, 2, 3$, and zero otherwise. Does there exist a closed orientable 3-manifold with these homology groups?

2. A space X is constructed by gluing up the solid hexagonal prism shown below: the hexagonal faces are glued using translation and a 60 degree rotation, and the opposite sides of the prism are glued in pairs via translation. Calculate the integral homology of X.

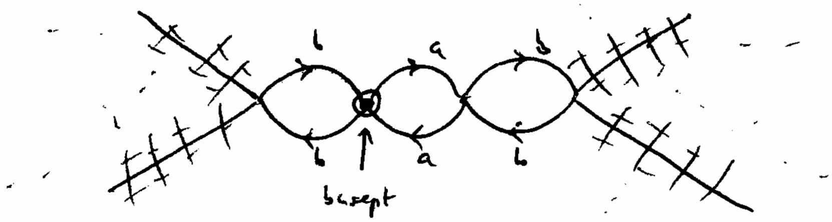


3. Recall that for any $p \geq 2$, the 3-dimensional lens space $L^3(p,1)$ has integral homology groups $\mathbb{Z}, \mathbb{Z}_p, 0, \mathbb{Z}$ in dimensions $0, 1, 2, 3$. Calculate the integral homology of the product $L(p,1) \times L(q,1)$.

4. Let X_n be the bouquet of n circles, whose fundamental group (based at the vertex of the bouquet) is the free group F_n on n generators.

(i). Construct a basepointed covering of X_3 corresponding to the subgroup $< b^3, a^2, b^2ab^{-1} >$ of the free group $< a, b, c >$.

(ii). Find the subgroup of F_2 corresponding to the basepointed cover of X_2 depicted below.



5. Show that if M is a compact orientable manifold with boundary ∂M, then there does not exist a retraction $r : M \to \partial M$.

6. Show that any homotopy equivalence from $\mathbb{C}P^{2n}$ to itself is orientation-preserving, that is has degree $+1$.

7. Suppose X is a 1-connected CW complex whose homology groups are $\mathbb{Z}$ in dimension 0, $\mathbb{Z}^2$ in dimension 3, and zero otherwise. By constructing a map $S^3 \vee S^3 \to X$, show that X is homotopy equivalent to $S^3 \vee S^3$.

8. Let M^{2n} be a closed orientable even-dimensional manifold. Show that its Euler characteristic is odd if and only if the dimension of $H_n(M; \mathbb{Q})$ is odd, and that consequently a closed manifold of dimension $4n+2$ with odd Euler characteristic must be non-orientable.