2001 Topology 290 – Qualifying exam

Do all questions; each is worth 10 marks. The exam lasts 3 hours.

Standard theorems may be assumed as long as you make clear when you are using them. Please try to write good, readable mathematics: marks will be deducted for poor organisation, unclear logic, and anything else that impedes comprehension!

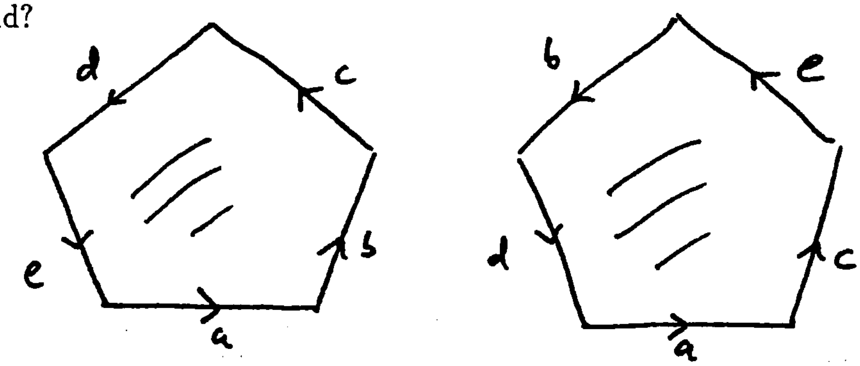
1. Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ (where $a_n \neq 0$, and $n \geq 1$) be a complex polynomial. Prove by a topological argument that p must have a root in the complex plane.

2. Let Σ_g be a closed orientable surface of genus g. A map $\pi : \Sigma_g \to S^2$ is a *double branched cover* if there is a set $Q = \{p_1, p_2, \ldots, p_n\} \subseteq S^2$ of *branch points*, so that π restricted to $\Sigma_g - \pi^{-1}(Q)$ is a double cover of $S^2 - Q$, but the points p_i have only one preimage each. Use Euler characteristic to find a formula relating g and n.

3. The *connect-sum* (#) of two oriented 4-manifolds is defined by removing an open 4-ball from each, and gluing the resulting manifolds using a homeomorphism between their boundary 3-spheres, in such a way that the orientations match to make a new oriented manifold. Compute the cohomology ring of the connect-sum $X = \mathbb{C}P^2 \# (S^2 \times S^2)$.

4. Let X be a (path-connected) simply-connected CW-complex with $H_2(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $H_{\geq 3}(X) = 0$. Prove that X is homotopy-equivalent to the "bouquet of two spheres" $S^2 \vee S^2$.

5. Let X be the result of gluing up the edges of two solid pentagons in pairs, according to the picture shown below. Compute the fundamental group and the homology groups of X. Is it a manifold?



6. Show that any homotopy equivalence from $\mathbb{C}P^{2n}$ to itself is orientation-preserving, i.e. has degree +1. Is this true for $\mathbb{C}P^{2n+1}$?