Topology Qualifying Exam

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(1) Let X be a CW-complex obtained from $S^n, n \geq 1$, by attaching a single $(n+1)$-cell by a map $\varphi : S^n \to S^n$ of degree 4. Compute the cohomology ring $H^*(X;\mathbb{Z}/2)$.

(2) Let $E \to B$ be a d-sheeted covering map. Prove that the Euler characteristic satisfies $e(E) = d \cdot e(B)$ if B is a finite CW-complex.

(3) Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Show that for $n \geq 1$ the Hopf fibrations

$$p : \mathbb{K}^{n+1} \setminus 0 \longrightarrow \mathbb{KP}^n$$

do *not* admit a section s (i.e. there is *no* continuous s satisfying $p \circ s = id$).

(4) Consider the polynomial ring $R := \mathbb{Q}[x,y]$ and make $\mathbb{Q}$ an R-module via the zero actions of x and y. Compute for all i

$$\mathrm{Tor}_i^R(\mathbb{Q},\mathbb{Q})$$

and show that every R-module has a free resolution of length 2 (one longer than a PID).

(5) Using the fundamental class in $\mathbb{Z}/2$-homology, there is an obvious definition of the *mod 2-degree* for a continuous map between closed manifolds (not necessarily oriented).

Show that there is *no* map of nonzero mod 2-degree between the Klein bottle and the torus, in either direction.

(6) Show that $\mathbb{RP}^2$ is *not* the boundary of a compact 3-manifold.

Do you think that $\mathbb{RP}^3$ is the boundary of a compact 4-manifold?