Topology Qualifying Exam 2005

Throughout the exam we assume all spaces have finitely generated homology. All subspaces will have collared neighborhoods. All spaces will be connected, locally path connected, and semilocally simply connected.

1. (a) Let G be a group of homeomorphisms of S^{2n} such that for all $g \in G$, $gx = x$ if and only if $g = 1$. Prove if G has 2 elements, then one of them is the antipodal map.

   (b) If $S^{2n} \longrightarrow X$ is a covering space, prove X is homeomorphic to S^{2n} or $\mathbb{RP}^{2n}$.

2. Let X be a compact $\mathbb{Z}_k$ orientable manifold for $k > 2$. Prove X is orientable.

3. (a) Compute the integral and mod 2 cohomology and homology of $\mathbb{RP}^2 \times \mathbb{RP}^3$. Is $\mathbb{RP}^2 \times \mathbb{RP}^3$ orientable?

   (b) Determine the action of the Bockstein
$$\beta_1 : H^*(\mathbb{RP}^2 \times \mathbb{RP}^3; \mathbb{Z}_2) \longrightarrow H^{*+1}(\mathbb{RP}^2 \times \mathbb{RP}^3; \mathbb{Z}_2).$$

4. Determine all covering spaces of $S^1 \times \mathbb{RP}^3$. (This includes determining the covering projections).

5. Let X be a compact non-orientable 3-manifold. Prove $H^1(X; \mathbb{Z}) \neq 0$.

6. Let X be a compact space having the homotopy type of $S^3 \vee S^5$. Determine if X can be a manifold or an H-space.

7. Let $f : S^{2k+1} \longrightarrow S^{2k+1}$ satisfy $f(-x) = -f(x)$. Prove degree of f is odd.

8. Show there is no map $f : \mathbb{HP}^n \longrightarrow \mathbb{CP}^{2n}$ such that the induced map $H_{4n}(f) : H_{4n}(\mathbb{HP}^n; \mathbb{Z}) \longrightarrow H_{4n}(\mathbb{CP}^{2n}; \mathbb{Z})$ maps the generator $\zeta_{\mathbb{HP}^n}$ to $k\zeta_{\mathbb{CP}^{2n}}$ for $k \neq 0$.

9. Let p be an odd prime. Recall the lens space $L(p, q)$ is a 3-dimensional compact manifold with
$$H_\ell(L(p,q); \mathbb{Z}) = \begin{cases} \mathbb{Z} & \ell = 0, 3 \\ \mathbb{Z}_p & \ell = 1 \\ 0 & \ell = 2 \end{cases}.$$
Construct a space X with $H_*(X; \mathbb{Z}) = H_*(L(p,q); \mathbb{Z})$ but X is not homotopy equivalent to $L(p,q)$. Verify that X is not homotopy equivalent to $L(p,q)$.

10. Let $X \vee Y$ be the one point union of $X \times Y$. Prove for each $\ell > 0$ there is a split short exact sequence
$$0 \longrightarrow H_\ell(X \vee Y) \longrightarrow H_\ell(X \times Y) \longrightarrow H_\ell(X \times Y, X \vee Y) \longrightarrow 0$$