45. Summer 2025

Three-hour exam. Do as many questions as you can. Each is worth 4 marks. Please write clear maths and clear English which could be understood by one of your fellow students - pictures aid explanation but should not replace it! Include as much detail as is appropriate; you can use standard results and theorems in your answers provided you refer to them clearly. Notes may not be used.

1. Let X be the result of deleting 10 points from the 2-sphere. Compute its fundamental group. (You may assume whatever you like about the points in order to simplify the calculation.)

2. Let $\mathbb{R}P^4$ be realised as a subspace of $\mathbb{R}P^9$ by setting the last 5 homogeneous coordinates equal to 0. Let X be the quotient $\mathbb{R}P^9/\mathbb{R}P^4$. Compute its singular homology with integral coefficients.

3. Let T be the standard torus, and let e and f be generators of $H_1(T;\mathbb{Z}) \cong \mathbb{Z}^2$. Now let X be the space obtained by gluing two discs onto T along their boundary circles: the first attaches along a curve with homology class $e + 4f$, and the second along a curve with homology class $4e + f$. Calculate the integral homology groups $H_*(X;\mathbb{Z})$.

4. For any topological space X whose total homology is a finitely-generated abelian group, let $\chi(X)$ denote the usual Euler characteristic

$$\chi(X) = \sum (-1)^i \dim_{\mathbb{Q}} H_i(X;\mathbb{Q})$$

and let $\chi_2(X)$ be the "mod-2 homology Euler characteristic"

$$\chi_2(X) = \sum (-1)^i \dim_{\mathbb{F}_2} H_i(X;\mathbb{F}_2).$$

Use the universal coefficient theorem to show that $\chi(X) = \chi_2(X)$.

5. Let $X = S^2 \times S^3$ and let $Y = S^2 \vee S^3 \vee S^5$. Explain why the spaces have isomorphic integral cohomology groups in each degree. Is X homotopy-equivalent to Y?

6. Show that any homotopy equivalence from $\mathbb{C}P^{2n}$ to itself is orientation-preserving, that is has degree $+1$.

7. Prove that there is no compact 4-manifold M (with or without boundary) which is homotopy-equivalent to the suspension $\Sigma\mathbb{R}P^3$.

8. Let P be the Poincaré homology sphere, a 3-manifold whose fundamental group has order 120 and whose universal cover is S^3. Compute π_3 of the wedge sum $P \vee S^3$.